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PISA 2009 Mathematics Framework

This chapter begins with PISA's definition of mathematics and explains the theory underlying the development of the assessment. The PISA mathematics assessment is organised into many different categories: situations or contexts; mathematical content and mathematical competencies. It also includes four overarching ideas: space and shape; change and relationships; quantity and uncertainty. The chapter describes the processes and competencies needed to solve PISA mathematics questions and explains the categorisation of the three competency clusters: reproduction, connections and reflections. It uses sample tasks from the PISA assessment to further illustrate the framework and then discusses how proficiency in mathematics is measured for the analysis and reporting of results.



INTRODUCTION

This framework describes and illustrates the PISA mathematics assessment. The term *mathematical literacy* is used to highlight that the PISA mathematics assessment is concerned with the reproduction of mathematical knowledge and in addition, in solving the PISA assessment tasks, students are typically required to extrapolate from what they have learned in school and to apply mathematical knowledge to authentic problems situated in a variety of contexts. The mathematical processes needed to do this, which are based on mathematical knowledge and skills, are referred to as cognitive mathematical competencies. The major components of the mathematics framework, consistent with the other PISA frameworks, include contexts for the use of mathematics, mathematical content and mathematical processes, each of which flow directly out of the definition of the domain. The discussions of context and content in this chapter emphasise features of the problems that confront students as citizens, while the discussions of processes emphasise the mathematical knowledge and skills that students employ to solve those problems. These processes have been grouped into three clusters to facilitate a rational treatment of the way complex cognitive processes are addressed within a structured assessment programme.

DEFINITION OF THE DOMAIN

The PISA *mathematics* domain is concerned with the ability of students to analyse, reason and communicate ideas effectively as they pose, formulate, solve and interpret mathematical problems in a variety of situations. The PISA mathematics assessment focuses on real-world problems, moving beyond the kinds of situations and problems typically encountered in school classrooms. In real-world settings, citizens routinely face situations in which the use of quantitative or spatial reasoning or other cognitive mathematical competencies would help clarify, formulate or solve a problem. Such situations include shopping, travelling, cooking, dealing with personal finances, judging political issues, etc. Such uses of mathematics are based on the skills learned and practised through the kinds of problems that typically appear in school textbooks and classrooms. However, they also demand the ability to apply those skills in a less structured context, where the directions are not so clear, and where the student must make decisions about what knowledge may be relevant and how it might be usefully applied.

Citizens in every country are increasingly confronted with a myriad of tasks involving quantitative, spatial, probabilistic and other mathematical concepts. For example, media outlets (newspapers, magazines, television and the Internet) are filled with information in the form of tables, charts and graphs about subjects such as weather, climate change, economics, population growth, medicine and sports, to name a few. Citizens are also confronted with the need to read forms, interpret bus and train timetables, successfully carry out transactions involving money, determine the best buy at the market, and so on. The PISA mathematics assessment focuses on the capacity of 15-year-old students (the age when many students are completing their formal compulsory mathematics learning) to use their mathematical knowledge and understanding to help make sense of these issues and carry out the resulting tasks.

PISA defines *mathematical literacy* as:

...an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgments and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen.

Some explanatory remarks may help to further clarify this domain definition:

- The term *mathematical literacy* emphasises mathematical knowledge put to functional use in a multitude of different situations in varied, reflective and insight-based ways. Of course, for such use to be possible and viable, many fundamental mathematical knowledge and skills are needed. Literacy in the linguistic sense presupposes, but cannot be reduced to, a rich vocabulary and substantial knowledge of grammatical rules, phonetics, orthography, etc. To communicate, humans combine these elements in creative ways in response to each real-world situation encountered. In the same way, mathematical literacy presupposes, but cannot be reduced to, knowledge of mathematical terminology, facts and procedures, as well as skills in performing certain operations and carrying out certain methods. It involves the creative combination of these elements in response to the demands imposed by external situations.



- The term *the world* refers to the natural, social and cultural setting in which the individual lives. As Freudenthal (1983) states: “Our mathematical concepts, structures, ideas have been invented as tools to organise the phenomena of the physical, social and mental world” (p. ix).
- The phrase *to use and engage with* is meant to cover the usage of mathematics and solving mathematical problems, and also implies a broader personal involvement through communicating, relating to, assessing, and even appreciating and enjoying mathematics. Thus, the definition of *mathematical literacy* encompasses the functional use of mathematics, in a narrow sense, as well as preparedness for further study and the aesthetic and recreational elements of mathematics.
- The phrase “that individual’s life” includes his or her private life, occupational life and social life with peers and relatives, as well as his or her life as a citizen of a community.

A crucial capacity implied by this notion of *mathematics* is the ability to pose, formulate, solve and interpret problems using mathematics within a variety of situations and contexts. The contexts range from being purely mathematical to having no mathematical structure present or apparent at the outset – the problem poser or solver must successfully introduce the mathematical structure. It is also important to emphasise that the definition is not just concerned with knowing mathematics at some minimal level; it is also about doing and using mathematics in situations that range from the everyday to the unusual, from the simple to the complex.

THEORETICAL BASIS FOR THE PISA MATHEMATICS FRAMEWORK

The PISA definition of *mathematical literacy* is consistent with the broad and integrative theory about the structure and use of language as reflected in recent socio-cultural literacy studies. In James Gee’s *Preamble to a Literacy Program* (1998), the term *literacy* refers to the human use of language. The ability to read, write, listen and speak a language is the most important tool through which human social activity is mediated. In fact, each human language and use of language has an intricate design tied in complex ways to a variety of functions. Being literate in a language implies that a person knows many of the design resources of the language and is able to use those resources for several different social functions. Analogously, considering mathematics as a language implies that students must learn the design features involved in mathematical discourse (the terms, facts, signs and symbols, procedures and skills to perform certain operations in specific mathematical sub-domains, and the structure of those ideas in each sub-domain), and they also must learn to use such ideas to solve non-routine problems in a variety of situations defined in terms of social functions. Note that the design features for mathematics include knowing the basic terms, procedures and concepts commonly taught in schools, and also involve knowing how these features are structured and used. Unfortunately, one can know a good deal about the design features of mathematics without knowing either their structure or how to use those features to solve problems. These scholarly notions involving the interplay of design features and functions that support the mathematics framework for PISA can be illustrated in the following example.

MATHEMATICS EXAMPLE 1: HEARTBEAT

For health reasons people should limit their efforts, for instance during sports, in order not to exceed a certain heartbeat frequency.

For years the relationship between a person’s recommended maximum heart rate and the person’s age was described by the following formula:

Recommended maximum heart rate = $220 - \text{age}$

Recent research showed that this formula should be modified slightly. The new formula is as follows:

Recommended maximum heart rate = $208 - (0.7 \times \text{age})$

The questions in this unit centre around the difference between the two formulae and how they affect the calculation of the maximum allowable heart rate.

Question 1: HEARTBEAT

A newspaper article stated: "A result of using the new formula instead of the old one is that the recommended maximum number of heartbeats per minute for young people decreases slightly and for old people it increases slightly."

From which age onwards does the recommended maximum heart rate increase as a result of the introduction of the new formula? Show your work.

Question 1 can be solved by following the general strategy used by mathematicians, which the mathematics framework will refer to as "mathematising". Mathematising can be characterised as having five aspects:

- *First, the process of mathematisation or mathematising starts with a problem situated in reality.*

As will be clear from the item, the issue situated in reality in this case is physical health and fitness: "An important rule when exercising is that one should be careful to not push oneself too far as excessive exertion may cause health problems." The question alerts us to this issue through the text linking health to heart rate and by referring to the "recommended maximum heart rate".

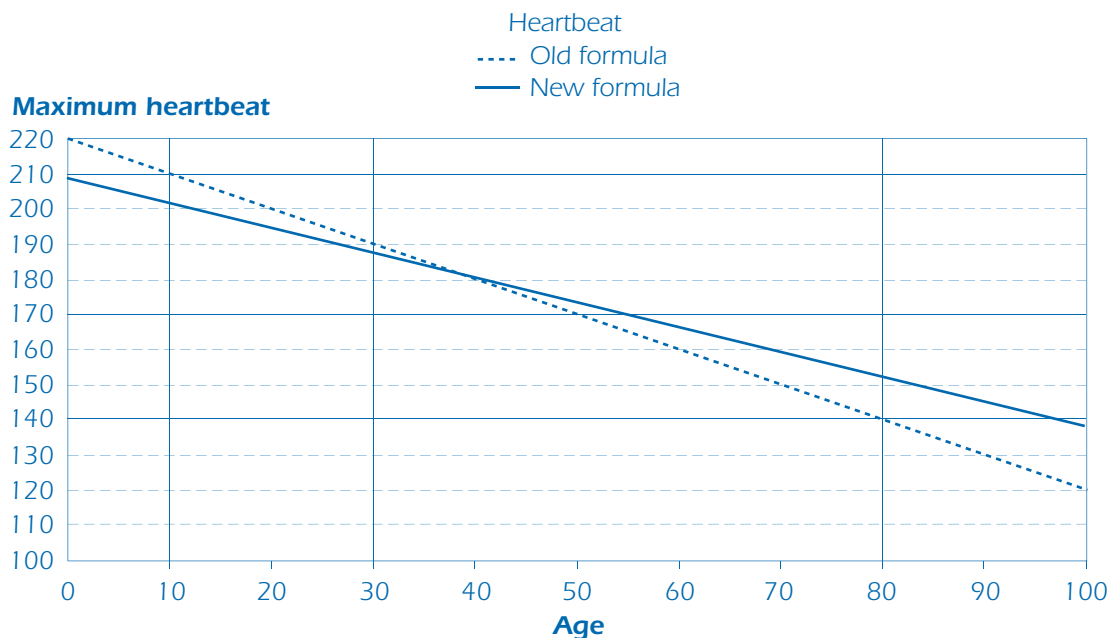
- *Secondly, the problem solver tries to identify the relevant mathematics, and reorganises the problem according to the mathematical concepts identified.*

It seems clear that the student faces two word formulae that need to be understood, and he or she is requested to compare these two formulae and determine what they mean in mathematical terms. The formulae give a relation between the advised maximum heart beat rate and the age of a person.

- *The third step involves gradually trimming away the reality.*

There are different ways of refocusing the problem to be a strictly mathematical problem, or of trimming away reality. One way would be to make the word formulae into more formal algebraic expressions like $y = 220 - x$ or $y = 208 - 0.7x$. The student must remember that y expresses the maximum heart beat in beats per minute and x represents the age in years. Another strictly mathematical approach would be to draw the graphs directly from the word formulae. These graphs are straight lines as the formulae are of the first degree. The graphs have different slopes, so they intersect.

These three steps lead one from a real-world problem to a mathematical problem.





■ *The fourth step is solving the mathematical problem.*

The mathematical problem at hand is to compare the two formulae, or graphs, and say something about the differences for people of a certain age. A nice way to start is to find out where the two formulae give equal results, or where the two graphs intersect. The student can find this by solving the equation: $220 - x = 208 - 0.7x$. This gives us $x = 40$ and the corresponding value for y is 180. So the two graphs intersect at the point (40, 180). This point can also be found in the graph just above. As the slope of the first formula is -1 and the second is -0.7, the student knows that the second graph is less steep than the first. The student also knows that the graph line of $y = 220 - x$ lies above the graph line of $y = 208 - 0.7x$ for values of x smaller than 40 and lies below for values of x larger than 40.

■ *The fifth asks what the meaning of this strictly mathematical solution is in terms of the real world.*

The meaning is not too difficult if the student realises that x is the age of a person and y the maximum heart beat. If one is 40 years old both formulae give the same result: a maximum heartbeat of 180. The 'old' rule allows for higher heart rates for younger people: in the extreme, if the age is zero the maximum is 220 in the old formula and only 208 in the new formula. But for older people, in this case for those over 40, the more recent insights allow for higher maximum heartbeat; as an example: for an age of 100 years the student sees that the old formula gives him or her a maximum of 120 and the new one 138. Of course the student has to realise a number of other things: the formulae lack mathematical precision and are only quasi scientific. In reality, the formulae provide only a rule of thumb that should be used with caution. Moreover the outcomes for ages at the extreme should be regarded even more cautiously.

What this example shows is that even with items that seem relatively simple in the sense that they can be used within the restrictions of a large international study and solved in a short time, the full cycle of mathematisation and problem solving can still be identified.

It is these processes that characterise how, in a broad sense, mathematicians often do mathematics, how people use mathematics in a variety of current and potential occupations, and how informed and reflective citizens should use mathematics to fully and competently engage with the real world. In fact, learning to mathematise should be a primary educational goal for all students.

Today and in the foreseeable future, every country needs mathematically literate citizens to deal with a very complex and rapidly changing society. Accessible information has been growing exponentially and citizens need to be able to decide how to deal with this information. Social debates increasingly involve quantitative information to support claims. Individuals often need to be mathematically literate to make judgements and assess the accuracy of conclusions and claims in surveys and studies. Being able to judge the soundness of the claims from such arguments is, and increasingly will be, a critical aspect of being a responsible citizen. The steps of the mathematisation process discussed in this framework are fundamental for one to use mathematics in such complex situations. Failure to use mathematical notions can result in confused personal decisions, an increased susceptibility to pseudo-sciences, and poorly informed decision-making in professional and public life.

A mathematically literate citizen understands how quickly change is taking place and the consequent need to be open to lifelong learning. Adapting to these changes in a creative, flexible and practical way is a necessary condition for successful citizenship. The skills learned at school will probably not be sufficient to serve the needs of citizens for the majority of their adult life.

The requirements for competent and reflective citizenship also affect the workforce. Workers are less frequently expected to carry out repetitive physical chores. Instead, they are engaged actively to monitor output from a variety of high-technology machines, deal with a flood of information and engage in team problem solving. In the future more occupations will require the ability to understand, communicate, use and explain concepts and procedures based on mathematical thinking. The steps of the mathematisation process are the building blocks of this kind of mathematical thinking.

Mathematically literate citizens also develop an appreciation for mathematics as a dynamic, changing and relevant discipline that may often serve their needs.

The operational problem faced by PISA is how to assess whether 15-year-old students are mathematically literate in terms of their ability to mathematise. Unfortunately, in a timed assessment this is difficult because most complex real situations demand a considerable amount of time for one to collaborate and find appropriate resources while proceeding from reality to mathematics and back.

To illustrate mathematisation in an extended problem-solving-exercise, consider the following HOLIDAY example, which was an item in the PISA 2003 problem-solving survey. The problem poses two questions to the students. It deals with the planning of a route and places to stay overnight on a holiday trip. Students were presented with a simplified map and a chart (multiple representations) showing the distances between the towns illustrated on the map.

MATHEMATICS EXAMPLE 2: HOLIDAY

This problem is about planning the best route for a holiday.

Figures A and B show a map of the area and the distances between towns.

Figure A. Map of roads between towns

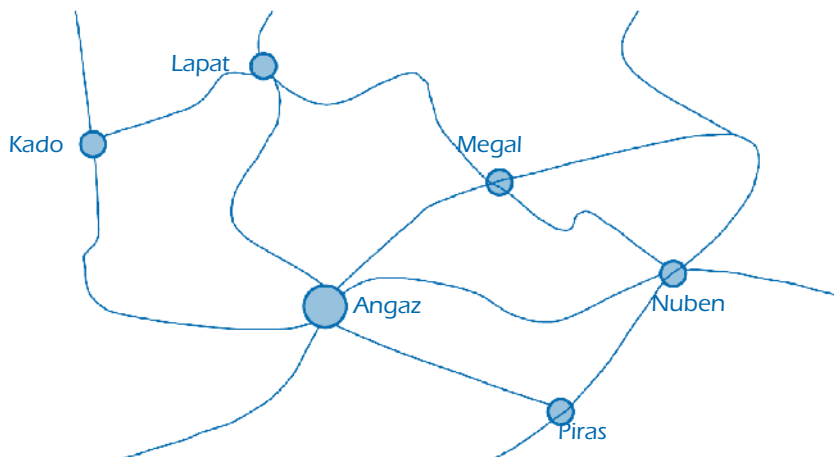


Figure B. Shortest road distance of towns from each other in kilometres

Angaz						
Kado	550					
Lapat	500	300				
Megal	300	850	550			
Nuben	500		1000	450		
Piras	300	850	800	600	250	
	Angaz	Kado	Lapat	Megal	Nuben	Piras

Question 1: HOLIDAY

Calculate the shortest distance by road between Nuben and Kado.

Distance: kilometres.



Question 2: HOLIDAY

Zoe lives in Angaz. She wants to visit Kado and Lapat. She can only travel **up to 300 kilometres** in any one day, but can break her journey by camping overnight anywhere between towns.

Zoe will stay for two nights in each town, so that she can spend one whole day sightseeing in each town.

Show Zoe's itinerary by completing the following table to indicate where she stays each night.

Day	Overnight Stay
1	Camp-site between Angaz and Kado
2	
3	
4	
5	
6	
7	Angaz

While there is no obvious link to a curricular discipline, there is a clear relation to discrete mathematics. There is also not a pre-described strategy to solve this problem. Often, students know exactly which strategy to use when posed a problem-solving exercise. But in real-world problem solving there is no well-known strategy available.

This example also clearly presents the five aspects of mathematising. The problem is situated in reality and can be organised according to mathematical concepts (distance tables or matrices) and maps (as models of reality.) Additionally, the student is required to trim away redundant information and focus on relevant information, especially on the mathematical aspects of that information. Finally, the student needs to solve the problem in mathematical terms then reflect on the solution in terms of the real situation.

Although there is relatively little reading required to solve the problem, it is still rather complex because students must read and interpret information from the map and the distance chart. Some of the distances that they have to find in the chart require them to read distances starting from the bottom of the chart and some starting from the left. For example, in determining the distance from Nuben to Piras, one needs to transform the search to that of finding the distance from Piras to Nuben (OECD, 2004).

The second question sets a number of constraints that needed to be complied with simultaneously: a maximum of 300 kilometres travelled in a given day, starting and finishing in Zoe's hometown of Angaz, visiting Kado and Lapat, and staying two nights in each of these cities so that she can achieve her vacation goals.

It should be noted that in the PISA problem-solving survey from which this item was taken, students were allowed considerably more time to find answers than normally allowed for the typically shorter mathematics items.

Ideally, to judge whether 15-year-old students can use their accumulated mathematical knowledge to solve mathematical problems they encounter in their world, one would collect information about the students' abilities



to mathematise such complex situations. Since this is not a feasible option though, PISA instead prepares items to assess different parts of this process. The following section describes how a set of test items are created in a balanced manner so that all five aspects of mathematising are covered in the selection. The aim is to use the responses to those items to locate students on a scale of proficiency in the PISA assessment of mathematics

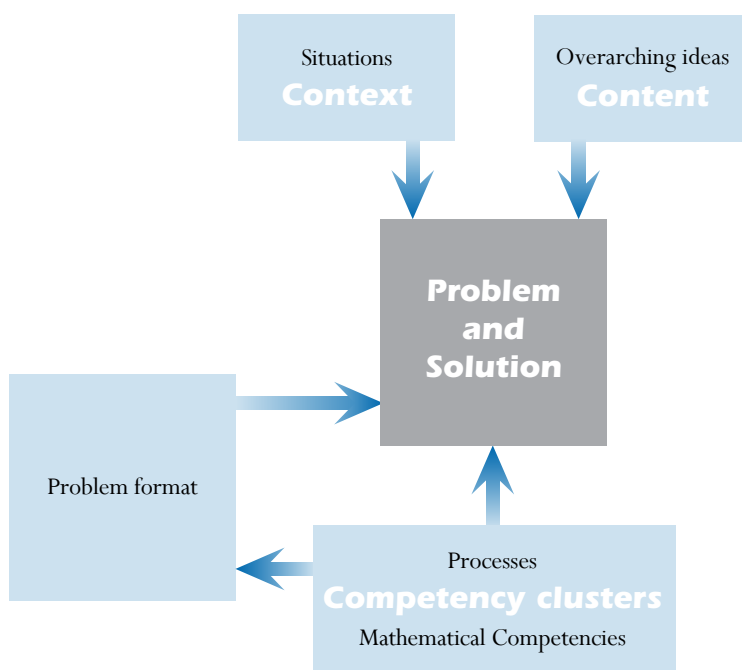
ORGANISATION OF THE DOMAIN

The PISA mathematics framework describes and provides the rationale for an assessment that examines how 15-year-old students handle mathematics in a well-founded manner when confronted with real-world problems, or more generally an assessment of how mathematically literate 15-year-old students are. To describe the domain more clearly, three components must be distinguished:

- the *situations or contexts* in which the problems are located;
- the *mathematical content* used to solve the problems, organised by certain overarching ideas;
- the *mathematical competencies* that must be activated to connect the real world, in which the problems are generated, with mathematics, and thus to solve the problems.

These components are represented in Figure 2.1. An explanation of each is provided afterwards.

Figure 2.1 ■ The components of the mathematics domain



A person's *mathematical literacy* is seen in the way he or she uses mathematical knowledge and skills to solve problems. Problems (and their solutions) may occur in a variety of situations or contexts within the experience of an individual. PISA problems draw from the real world in two ways. First, problems exist within some broad situations that are relevant to the student's life. The situations form part of the real world and are indicated by a big square in the upper left-hand corner of Figure 2.1. Next, within that situation, problems have a more specific context. This is represented by the grey rectangle in the situations square.

In the *HEARTBEAT* and *HOLIDAY* examples, both situations are the personal real world, and the contexts are sport/health aspects for the active citizen and planning a holiday.

To solve real world problems such as these, a person must apply his or her knowledge of specific technical



mathematical content. Unlike a classroom situation, where the mathematical content under study is usually evident, typically the knowledge required when solving real-world problems is not immediately evident. The problem solver must engage with the phenomena through which the problem is presented, must identify what specific knowledge will likely be useful, and must activate that knowledge.

PISA therefore identifies mathematical content by listing a small set of *overarching ideas* that represent broad categories of real-world phenomena through which opportunities to explore and use mathematics arise in our interactions with the world. This approach reflects the main threads in both the historical development of mathematics as a discipline, and the development of mathematical ideas in individuals. Those four broad categories are also elaborated to show the more specific mathematical objects, knowledge and skills typically used in relation to each, thereby showing the links between this approach to specifying mathematical content, and the topics covered in school mathematics curricula.

The *overarching ideas* are represented by the big square in the upper right-hand corner of Figure 2.1. From the overarching ideas, the particular mathematical content used in solving a problem is extracted. This is represented by the smaller square within the overarching ideas square.

The arrows going from the context and content to the problem show how the real world (including mathematics) makes up a problem.

The *HEARTBEAT* problem involves mathematical relations and comparing two relations in order to make decisions. Thus, the problem belongs to the overarching idea *change and relationships*. The *HOLIDAY* problem requires some basic computation, and while the second question requires some analytic reasoning, the most appropriate overarching idea is *quantity*.

Cognitive mathematical competencies are the mathematical processes that students apply as they attempt to solve problems. They encapsulate the different cognitive processes needed to solve various kinds of problems. The individual mathematical competencies as defined in the competency clusters reflect the way that mathematical processes are typically employed when solving problems that arise as students interact with their world. They will be described in detail in later sections.

Thus the process component of this framework is represented first by the large square, representing the general mathematical competencies, and, inside that, a smaller square that represents the three clusters. The mathematical processes or competencies needed to solve a particular problem are related to the nature of the problem, and the competencies used will be reflected in the solution found. This interaction is represented by the arrow from the box containing the mathematical processes to the box containing the problem and its solution.

The remaining arrow in the diagram goes from the box containing the mathematical processes to the box containing the problem format. The mathematical processes used to solve a problem are related to the form of the problem and its precise demands.

It should be emphasised that the three components just described are of different natures. Indeed, the mathematical processes are the core of the mathematics assessment, and students will only be in a position to successfully solve problems when certain competencies are available to them. Assessing *mathematical literacy* includes assessing to what extent students possess mathematical knowledge and skills that they can productively apply in problem situations.

In the following sections, these three components are described in more detail.

Situations and context

An important aspect of *mathematical literacy* is engagement with mathematics: using and doing mathematics in a variety of situations. It has been recognised that in dealing with issues that call for mathematical treatment, the choice of mathematical methods and representations is often dependent on the situations in which the problems are presented.



The situation is the part of the student's world in which the tasks are placed. It is located at a certain distance from the students. For PISA, the closest situation is the student's personal life. Next is school life, work life and leisure, followed by the local community and society as encountered in daily life. Scientific situations are furthest away. The four situation types defined and used for problems to be solved are: personal, educational/occupational, public and scientific.

The context of an item is its specific setting within a situation. It includes all the detailed elements used to formulate the problem.

Consider the following example:

MATHEMATICS EXAMPLE 3: SAVINGS ACCOUNT

Question 1: SAVINGS ACCOUNT

1 000 zed is put into a savings account at a bank. There are two choices: one can get an annual rate of 4% OR one can get an immediate 10 zed bonus from the bank, and a 3% annual rate.

Which option is better after one year? After two years?

The situation of this item is finance and banking, a situation from the local community and society that PISA would classify as public. The context of this item concerns money (zeds) and interest rates for a bank account.

This kind of problem is one that could be part of the actual experience or practice of the participant in some real-world setting. It provides an authentic context for the use of mathematics, since the application of mathematics in this context would be genuinely directed to solving the problem. This can be contrasted with problems frequently seen in school mathematics texts, where the main purpose is to practise the mathematics involved rather than to use mathematics to solve a real-world problem. This authenticity in the use of mathematics is an important aspect of the design and analysis of items for PISA, strongly related to the definition of *mathematical literacy*.

It should be noted that this use of the term *authentic* is not intended to indicate that mathematics items are genuine or real. PISA mathematics uses the term *authentic* to indicate that the use of mathematics is genuinely directed to solving the problem at hand, rather than the problem being merely a vehicle for the purpose of practising some mathematics.

One should also note that some elements of the problem are made up, namely the money involved, the zeds. This fictitious element is introduced to ensure that students from certain countries are not given an unfair advantage.

The situation and context of a problem can also be considered in terms of the distance between the problem and the mathematics involved. If a task refers only to mathematical objects, symbols or structures, and makes no reference to matters outside the mathematical world, the context of the task is considered to be intra-mathematical, and will be classified as belonging to the scientific situation type. A limited range of such tasks is included in PISA, where the close link between the problem and underlying mathematics is made explicit in the problem context. More typically, problems encountered in the day-to-day experience of the student are not stated in explicit mathematical terms. They refer to real-world objects. These task contexts are extra-mathematical and the student must translate these problem contexts into a mathematical form. Generally speaking, PISA puts an emphasis on tasks that might be encountered in some real-world situations and possess an authentic context for the use of mathematics that influences the solution and its interpretation. This, however, does not preclude the inclusion of tasks in which the context is hypothetical, as long as it has some real elements, is not too far removed from a real-world situation, and requires an authentic use of mathematics to solve the problem. Example 4 shows a problem with a hypothetical context that is extra-mathematical.



MATHEMATICS EXAMPLE 4: COINAGE SYSTEM

Question 1: COINAGE SYSTEM

Would it be possible to establish a coinage system based on only the denominations 3 and 5? More specifically, what amounts could be reached on that basis? Would such a system be desirable?

This problem does not necessarily derive its quality from its closeness to the real world; rather, it is mathematically interesting and calls on mathematical processes that are related to *mathematical literacy*. The use of mathematics to explain hypothetical scenarios and explore potential systems or situations is one of the most powerful features of this example, even if its actual scenarios or systems are unlikely to be carried out in reality. Such a problem would be classified as belonging to the scientific situation type.

In summary, PISA places most value on tasks that could be encountered in a variety of real-world situations and have a context in which the use of mathematics to solve the problem would be authentic. Problems with extra-mathematical contexts that influence the solution and its interpretation are preferred as a vehicle for assessing mathematics since these problems are most like those encountered in day-to-day life.

Mathematical content – the four overarching ideas

Mathematical concepts, structures and ideas have been invented as tools to understand, organise, and analyse the phenomena of the natural, social and mental world. In schools, the mathematics curriculum has been logically organised around content strands (e.g., arithmetic, algebra, geometry) and their detailed topics that reflect historically well-established branches of mathematical thinking, and that facilitate the development of a structured teaching syllabus. However, in the real world the phenomena that lend themselves to mathematical treatment do not come so logically organised. Rarely do problems arise in ways and contexts that allow their understanding and solution to be achieved through the application of knowledge from a single content strand, and solving problems as they appear in the real world usually requires an expanded range of thought processes compared with those typically employed in the classroom.

Since the goal of PISA is to assess students' capacity to solve real problems, our strategy has been to define the range of content that will be assessed using a phenomenological approach to describing mathematical concepts, structures or ideas. This means describing content in relation to the phenomena and the kinds of problems for which it was created. This approach ensures a focus in the assessment that is consistent with the domain definition, yet covers a range of content that includes what is typically found in other mathematics assessments and in national mathematics curricula.

A phenomenological organisation for mathematical content is not new. Two well known publications *On the Shoulders of Giants: New Approaches to Numeracy* (Steen, 1990) and *Mathematics: The Science of Patterns* (Devlin, 1994) have described mathematics in this manner. Various terms and categorisations have been used in these and other similar publications. A choice is needed, and for PISA four have been chosen. The overarching ideas used for PISA assessment purposes reflect an understanding of the mathematics field that focuses on patterns. Patterns in *quantity*, patterns in *space and shape* and patterns in *change and relationships* form central and essential concepts for any description of mathematics, and they form the heart of any curriculum, whether at high school, college or university. But increasingly, dealing with uncertainty from a mathematical and scientific perspective is also seen to be essential. For this reason, elements of probability theory and statistics give rise to the fourth overarching idea, *uncertainty*.

While this approach is somewhat different from the approach to content that would be taken from the perspective of mathematics instruction and the curricular strands typically taught in schools, nevertheless these overarching ideas encompass the full range of mathematical topics that students are expected to learn during their school mathematics studies.



The following list of overarching ideas, therefore, is used in PISA to meet the requirements of historical development, coverage of the domain and reflection of the major threads of school curriculum:

- space and shape
- change and relationships
- quantity
- uncertainty

With these four overarching ideas, mathematical content can be organised into a sufficient number of areas that ensure a spread of items across the curriculum while still maintaining a sufficiently broad focus to facilitate the presentation of problems based in real situations.

The basic conception of an overarching idea is an encompassing set of phenomena and concepts that make sense and can be encountered within and across a multitude of different situations. Each overarching idea can be perceived as a sort of general notion dealing with some generalised content dimension. This implies that neither can the overarching ideas, nor traditional mathematics content strands be sharply delineated in relation to one another. Rather, each idea represents a certain perspective or point of view and can be thought of as possessing a core, a centre of gravity, and somewhat blurred outskirts that allow for intersection with other overarching ideas. In principle, any overarching idea intersects any other overarching idea. The four overarching ideas are described in the following section.

Space and shape

Patterns are encountered everywhere: in spoken words, music, video, traffic, building constructions and art. Shapes can be regarded as patterns: of houses, office buildings, bridges, starfish, snowflakes, town plans, clover leaves, crystals or shadows. Geometric patterns can serve as relatively simple models of many kinds of phenomena, and their study is possible and desirable at all levels (Grünbaum, 1985).

It is important to have an understanding of the properties of objects and their relative positions. Students must be aware of how and why they see things and must learn to navigate through space and through constructions and shapes. This denotes an understanding of the relationship between shapes and images or visual representations, such as that between a real city and photographs and maps of the same city. It also includes an understanding of how three-dimensional objects can be represented in two dimensions, how shadows are formed and must be interpreted, and what perspective is and how it functions.

Shape has strong ties to traditional geometry, but goes far beyond it in content, meaning and method. Interaction with real shapes involves understanding the visual world and its description and encoding and decoding visual information. It also means interpreting visual information. In order to grasp the concept of shape, students should be able to discover the ways in which objects are similar and different, analyse the different components of an object, and recognise shapes in different dimensions and representations.

It is important to note that shapes can be more than just static entities. They can be transformed as entities, modified or sometimes visualised very elegantly using computer technology. Students should be able to see the patterns and regularities when shapes are changing. An example is shown in the following section in Figure 2.2.

The study of shapes and constructions requires looking for similarities and differences when analysing the components of form and recognising shapes in different representations and dimensions. The study of shapes is closely connected to the concept of *grasping space* (Freudenthal, 1973).

Examples requiring this kind of thinking are abundant and include the following: identifying and relating a photograph of a city to a map of that city and indicating from which point a picture was taken; the ability to draw a map; understanding why a nearby building looks bigger than a building that is further away; or understanding how the rails of a railway track appear to meet at the horizon. All these examples are relevant for students within this overarching idea.

As students live in a three-dimensional space, they should be familiar with views of objects from three orthogonal



aspects (for example, from the front, the side and above). They should be aware of the power and limitations of different representations of three-dimensional shapes, as indicated by the example provided in Figure 2.3. Students must not only understand the relative position of objects, but also how they can navigate through space and through constructions and shapes. An example is reading and interpreting a map and designing instructions on how to get from point A to point B using coordinates, common language or a picture.

Conceptual understanding of shapes also includes the ability to take a three-dimensional object and make a two-dimensional net of it, and vice-versa, even if the three-dimensional object is presented in two dimensions. An example of this is given in Figure 2.4.

Key aspects of *space and shape* are:

- recognising shapes and patterns in shapes;
- describing, encoding and decoding visual information;
- understanding dynamic changes to shapes;
- identifying similarities and differences;
- identifying relative positions;
- interpreting two-dimensional and three-dimensional representations and the relations between them;
- navigation through space.

Space and shape examples

Figure 2.2 offers a simple example of the need for flexibility in seeing shapes as they change. It is based on a cube that is being 'sectioned' (that is, plane cuts are made through the cube), and allows a variety of questions to be asked, such as:

What shapes can be produced by one plane cut through a cube?

How many faces, edges, or vertices will be produced when a cube is sectioned in this way?

Figure 2.2 ■ A cube, with plane cuts in various places



Three examples of the need for familiarity with representations of three-dimensional shapes follow. In the first example, the side and front views of an object constructed of cubes are given in Figure 2.3. The question is:

How many cubes have been used to make this object?

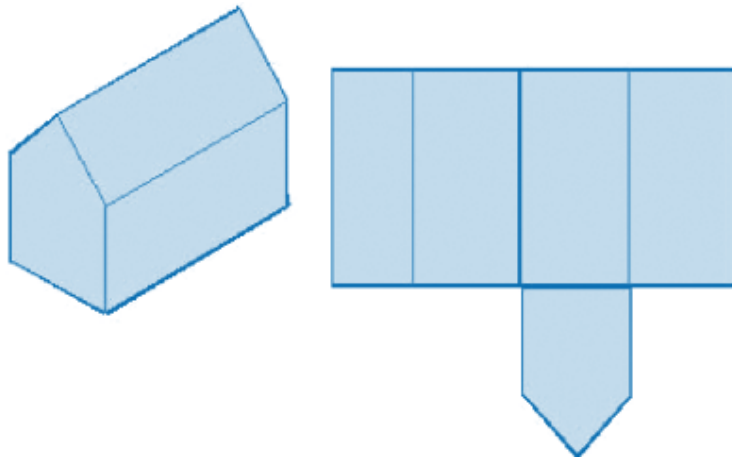
Figure 2.3 ■ Side and front views of an object made from cubes



It may come as a surprise to many – students and teachers alike – that the maximum number of cubes is 20 and the minimum is 6 (de Lange, 1995).

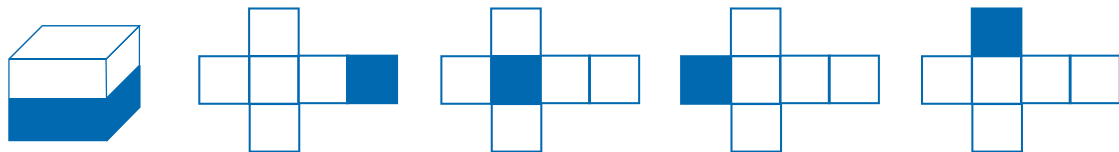
The next example shows a two-dimensional representation of a barn and an incomplete net of the barn. The problem is to complete the net of the barn.

Figure 2.4 ■ Two-dimensional representation of a three-dimensional barn and its (incomplete) net



A final example similar to the previous one is shown in Figure 2.5 (adapted from Hershkovitz *et al.*, 1996).

Figure 2.5 ■ Cube with black bottom



The lower half of the cube has been painted black. For each of the four nets, the bottom side is already black. Students could be asked to finish each net by shading the right squares.

Change and relationships

Every natural phenomenon is a manifestation of change, and the world around us displays a multitude of temporary and permanent relationships among phenomena. Examples include organisms changing as they grow; the cycle of seasons; the ebb and flow of tides; cycles of unemployment; weather changes; and stock exchange indices. Some of these processes of change involve and can be described or modelled by straightforward mathematical functions: linear, exponential, periodic or logistic, with any of these being either discrete or continuous. But many relationships fall into different categories, and data analysis is often essential in determining the kind of relationship that is present. Mathematical relationships often take the shape of equations or inequalities, but relations of a more general nature (e.g. equivalence, divisibility, inclusion, to mention but a few) may appear as well.

In order to be sensitive to the patterns of change, Stewart (1990) recommends:

- representing changes in a comprehensible form;
- understanding the fundamental types of change;
- recognising particular types of change when they occur;

- applying these techniques to the outside world;
- controlling a changing universe to the best advantage.



Change and relationships can be represented in a variety of ways including numerical (for example, in a table), symbolic, graphical, algebraic and geometrical. Translation between these representations is of key importance, as is the recognition of an understanding of fundamental relationships and types of change. Students should be aware of the concepts of linear growth (additive process), exponential growth (multiplicative process) and periodic growth, as well as logistic growth (at least informally as a special case of exponential growth).

Students should also see the relationships among these models – the key differences between linear and exponential processes, the fact that percentage growth is a form of exponential growth or how logistic growth occurs and why, either in continuous or discrete situations.

Changes occur in a system of interrelated objects or phenomena when the elements influence each other. In the examples mentioned in the summary, all phenomena change over time. But there are many real-life examples of matters in which objects are interrelated in a multitude of ways. For example:

If the length of the string of a guitar is halved, the new tone is an octave higher than the original tone. The tone is therefore dependent on the string length.

When we deposit money into a bank account, we know that the account balance will depend on the size, frequency and number of deposits and withdrawals, and the interest rates.

Relationships lead to dependency. Dependency concerns the fact that properties and changes of certain mathematical objects may depend on or influence properties and changes of other mathematical objects. Mathematical relationships often take the form of equations or inequalities, but relations of a more general nature may appear as well.

Change and relationships involves functional thinking. Functional thinking – that is, thinking in terms of and about relationships – is one of the most fundamental disciplinary aims of the teaching of mathematics (MAA, 1923). For 15-year-old students, this includes having a notion of rate of change, gradients and steepness (although not necessarily in a formal way), and dependence of one variable on another. Students should be able to make judgements about how fast processes are taking place in a relative way as well.

This overarching idea closely relates to aspects of other overarching ideas. The study of patterns in numbers can lead to intriguing relationships such as the study of Fibonacci numbers or the Golden Ratio. The Golden Ratio is a concept that plays a role in geometry as well, thus relating closely to the overarching idea of *space and shape*. Many more examples of *change and relationships* can be found in *space and shape*, such as with the growth of an area in relation to the growth of a perimeter or diameter. Euclidean geometry also lends itself to the study of relationships. A well-known example is the relationship between the three sides of a triangle. If the lengths of two sides are known, the third is not determined but the interval in which it lies is known; the interval's end points are the absolute value of the difference between the other two sides, and their sum, respectively. Several other similar relationships exist for the various elements of a triangle.

Uncertainty lends itself to various problems that can be viewed from the perspective of *change and relationships*. For example, if two fair dice have been rolled and one of them shows four, what is the chance that the sum exceeds seven? The answer (50%) relies on the dependency of the probability at issue on the set of favourable outcomes. The required probability is the proportion of all such outcomes compared with all possible outcomes, which is a functional dependency.

*Change and relationships examples***MATHEMATICS EXAMPLE 5: SCHOOL EXCURSION**

A school class wants to rent a coach for an excursion, and three companies are contacted for information about prices.

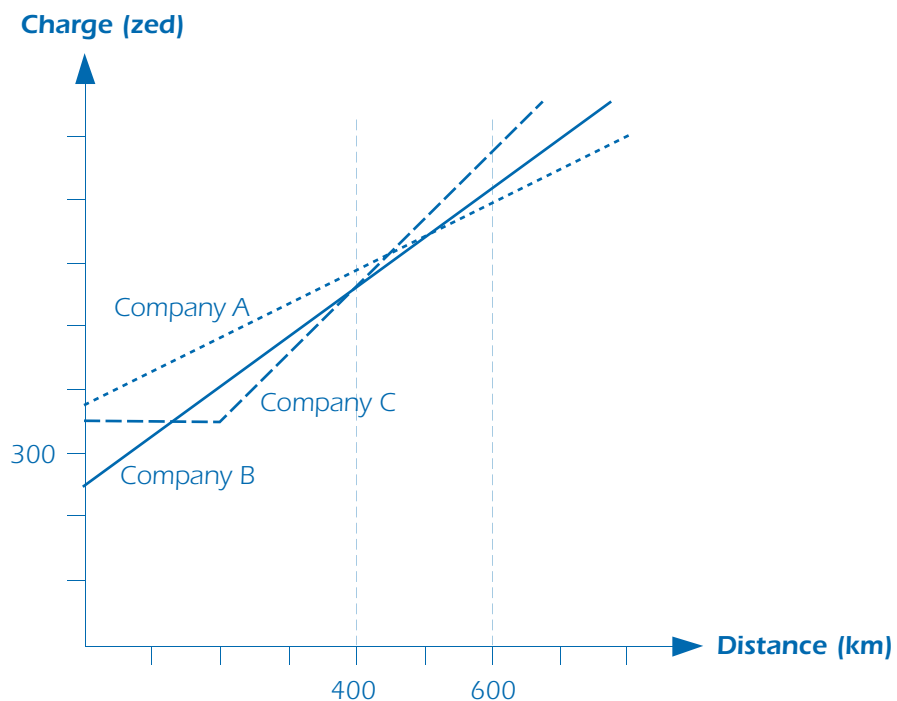
Company A charges an initial rate of 375 zed plus 0.5 zed per kilometre driven. Company B charges an initial rate of 250 zed plus 0.75 zed per kilometre driven. Company C charges a flat rate of 350 zed up to 200 kilometres, plus 1.02 zed per kilometre beyond 200 km.

Which company should the class choose, if the excursion involves a total travel distance of somewhere between 400 and 600 km?

Leaving aside the fictitious elements of the context, a problem like this could conceivably occur in the real world. Its solution requires the formulation and activation of several functional relationships, and equations and inequations. It can be dealt with by graphical as well as algebraic means, or combinations of both. The fact that the total travel distance in the excursion is not exactly known also introduces links to the *uncertainty* overarching idea, discussed in a later section.

A graphical representation of the problem is presented in Figure 2.6.

Figure 2.6 ■ Excursion charges for three bus companies



The following is another example of *change and relationships*.

MATHEMATICS EXAMPLE 6: CELL GROWTH

Doctors are monitoring the growth of cells. They are particularly interested in the day that the cell count will reach 60 000 because then they have to start an experiment. The table of results is:

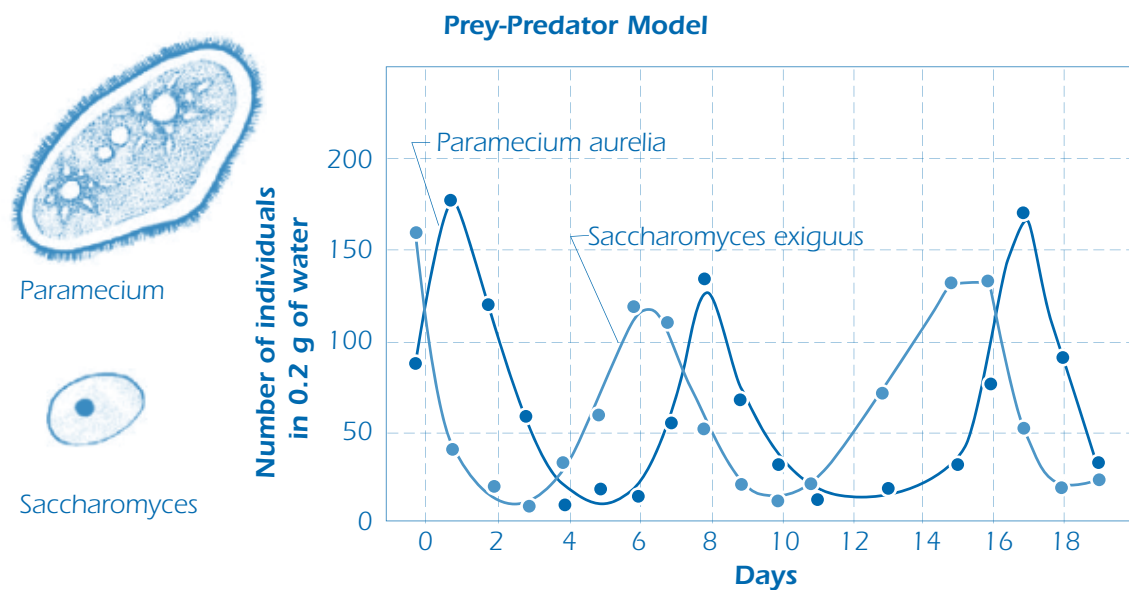
Time (days)	4	6	8	10	12	14	16	18	20	22	24
Cells	597	893	1 339	1 995	2 976	4 434	6 606	9 878	14 719	21 956	32 763

Question 1: CELL GROWTH

When will the number of cells reach 60 000?

MATHEMATICS EXAMPLE 7: PREY-PREDATOR

The following graph shows the growth of two living organisms: the Paramecium and Saccharomyces:



One of the two animals (predator) eats the other one (prey). Looking at the graph, can you judge which one is the prey and which one the predator?

One property of prey-predator phenomena is expressed as: The rate of growth of predators is proportional to the number of available prey. Does this property hold for the above graphs?

Quantity

Important aspects of *quantity* include an understanding of relative size, recognition of numerical patterns, and use of numbers to represent quantities and quantifiable attributes of real-world objects (counts and measures). *Quantity* additionally deals with the processing and understanding of numbers that are represented to us in various ways.

An important aspect of dealing with *quantity* is quantitative reasoning. Essential components of quantitative reasoning are number sense, representing numbers in various ways, understanding the meaning of operations, having a feel for the magnitude of numbers, mathematically elegant computations, mental arithmetic and estimating.

Some of the most important and frequent uses of numbers in every-day life are seen when magnitudes are measured: length, area, volume, height, speed, mass, air pressure, money value are all quantified using measures.



Understanding the meaning of operations includes the ability to perform operations involving comparisons, ratios and percentages. Number sense addresses issues of relative size, different representations of numbers, equivalent form of numbers and using understanding of these things to describe attributes of the world.

Quantity also includes having a feeling for quantities and estimation. In order to be able to test numerical results for reasonableness, one needs a broad knowledge of quantities (measures) in the real world. Is the average speed of a car 5, 50 or 500 km/h? Is the population of the world 6 million, 600 million, 6 billion, or 60 billion? How tall is a tower? How wide is a river? The ability to make quick order-of-magnitude approximations is of particular importance, especially when viewed in light of the increasing use of electronic calculating tools. One needs to be able to see that 33×613 is something around 20 000. To achieve this skill one does not need extensive training in mental execution of traditional written algorithms, but a flexible and smart application of place value understanding and single-digit arithmetic (Fey, 1990).

By using number sense in an appropriate way, students can solve problems requiring direct, inverse and joint proportional reasoning. They are able to estimate rates of change and provide a rationale for the selection of data and level of precision required by operations and models they use. They can also examine alternative algorithms, showing why they work or in what cases they fail. They can develop models involving operations, and relationships between operations, for problems involving real-world data and numerical relations requiring operations and comparisons (Dossey, 1997).

In the overarching idea of *quantity*, there is a place for ‘elegant’ quantitative reasoning like that used by Gauss, as discussed in the following example. Creativity coupled with conceptual understanding should be valued at the level of schooling that includes 15-year-old students.

Quantity examples

MATHEMATICS EXAMPLE 8: GAUSS

Karl Friedrich Gauss’ (1777–1855) teacher had asked the class to add together all the numbers from 1 to 100. Presumably the teacher’s aim was to keep the students occupied for a time. But Gauss was an excellent quantitative reasoner and spotted a short-cut to the solution. His reasoning went like this:

You write down the sum twice, once in ascending order, then in descending order, like this:

$$1 + 2 + 3 + \dots + 98 + 99 + 100$$

$$100 + 99 + 98 + \dots + 3 + 2 + 1$$

Now you add the two sums, column by column, to give:

$$101 + 101 + \dots + 101 + 101$$

As there are exactly 100 copies of the number 101 in this sum its value is: $100 \times 101 = 10\,100$.

Since this product is twice the answer to the original sum, if you halve it, you obtain the answer: 5 050.

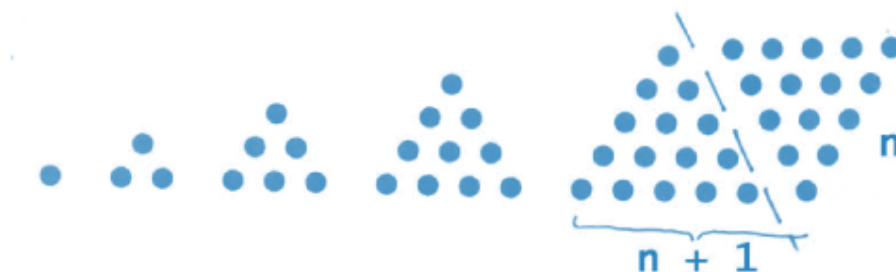
Triangular numbers

This example of quantitative thinking involving patterns of numbers can be taken a little further to demonstrate a link with a geometric representation of that pattern. The following formula gives the general situation for Gauss’s problem:

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

This formula also captures a geometric pattern that is well known: numbers of the form $\frac{n(n+1)}{2}$ are called triangular numbers since they are precisely the numbers that are obtained when balls are arranged in an equilateral triangle.

Figure 2.7 ■ The first five triangular numbers



The first five triangular numbers 1, 3, 6, 10 and 15 are shown in Figure 2.7.

Proportional reasoning

It will be interesting to see how students in different countries solve problems that lend themselves to the use of a variety of strategies. Differences can be expected especially in the area of proportional reasoning. In some countries, one strategy per item is likely to be used, while in other countries more strategies will be used. Also, similarities in reasoning will appear in solving problems that do not look very similar. This is in line with recent research results from TIMSS data (Mitchell, J. *et al.*, 2000). The following three items illustrate this point about different strategies and the relationships among them:

1. Tonight you're giving a party. You want to buy 100 cans of soft drink. How many six-can packs are you going to buy?
2. A hang-glider with glide-ratio 1 to 22 starts from a sheer cliff at 120 metres. The pilot is aiming at a spot at a distance of 1 400 metres. Will she reach that spot (under conditions of no wind)?
3. A school wants to rent mini-vans (with seats for eight passengers) for going to a school camp and 98 students need transportation. How many vans does the school need?

The first problem could be seen as a division problem ($100 \div 6 =$) that then leaves the student with an interpretation problem back to the context (what is the meaning of the remainder?). The second problem can be solved by proportional reasoning (for every metre height I can fly a distance of 22 metres, so starting from 120 metres...). The third problem will be solved by many as a division problem. All three problems, however, can be solved using the ratio table method:

Bottles :	1	10	5	15	2	17
	6	60	30	90	12	102
Flying :	1	100	20	120		
	22	2200	440	2640		
Buses :	1	10	2	13		
	8	80	16	104		

Identifying this similarity is a skill that belongs to mathematics: mathematically literate students do not need to look for the one available and appropriate tool or algorithm, but can choose from a wide array of strategies.





MATHEMATICS EXAMPLE 9: PERCENTS

Carl went to a store to buy a jacket with a normal price of 50 zed that was on sale for 20% off. In Zedland there is a 5% sales tax. The clerk first added the 5% tax to the price of the jacket and then took 20% off. Carl protested: he wanted the clerk to deduct the 20% discount first and then calculate the 5% tax.

Question 1: PERCENTS

Does it make any difference?

Problems involving this kind of quantitative thinking, and the need to carry out the resulting mental calculations, are encountered frequently when shopping. The ability to effectively handle such problems is fundamental to mathematics.

Uncertainty

Science and technology rarely deal with certainty. Indeed, scientific knowledge is seldom, if ever, absolute – and is even sometimes wrong – so some uncertainty always remains in even the most scientific predictions. Uncertainty is also present in daily life: uncertain election results, collapsing bridges, stock market crashes, unreliable weather forecasts, poor predictions for population growth, or economic models that don't align.

As an overarching idea, *uncertainty* suggests two related topics: data and chance. These phenomena are respectively the subject of mathematical study in statistics and probability. Relatively recent recommendations concerning school curricula are unanimous in suggesting that statistics and probability should occupy a much more prominent place than in the past (Committee of Inquiry into the Teaching of Mathematics in Schools, 1982; LOGSE, 1990; MSEB, 1990; NCTM, 1989; NCTM, 2000). Specific mathematical concepts and activities that are important in this area include collecting data, data analysis and display/visualisation, probability and inference.

The recommendations on the role of data, statistics and probability in school curricula emphasise data analysis. As a result, it is easy to view statistics in particular as a collection of specific skills. David S. Moore has pointed out what the overarching idea *uncertainty* really entails. The PISA definition will follow his ideas as presented in *On the Shoulders of Giants* (Steen, 1990) and F. James Rutherford's ideas as presented in *Why Numbers Count* (Steen, 1997).

Statistics brings something to mathematics education that is unique and important: reasoning from uncertain empirical data. This kind of statistical thinking should be part of the mental equipment of every intelligent citizen. The core elements are the:

- omnipresence of variation in processes;
- need for data about processes;
- design of data production with variation in mind;
- quantification of variation;
- explanation of variation.

Data are not merely numbers, but numbers in a context. Data are obtained by measurement and represented by a number. Thinking about measurement leads to a mature grasp of why some numbers are informative and others are irrelevant or nonsensical.

The design of sample surveys is a core topic in statistics. Data analysis emphasises understanding the specific data at hand, assuming they represent a larger population. The concept of simple random samples is essential for 15-year-old students to understand the issues related to uncertainty.

Phenomena have uncertain individual outcomes and the pattern of repeated outcomes is often random. The concept of probability in the present PISA study will generally be based to situations regarding chance devices like coins, number cubes and spinners, or not too complex real-world situations that can be analysed intuitively, or can feasibly be modelled with these devices.



Uncertainty also appears from sources like natural variation in students' heights, reading scores, incomes of a group of people, etc. A very important step, even for 15-year-old students, is to see the study of data and chance as a coherent whole. One such principle is the progression of ideas from simple data analysis to data production to probability to inference.

The important specific mathematical concepts and activities in this area are:

- producing data
- data analysis and data display/visualisation
- probability
- inference

Uncertainty examples

The following examples illustrate the *uncertainty* overarching idea.

MATHEMATICS EXAMPLE 10: AVERAGE AGE

If 40% of the population of a country are at least 60 years old, is it then possible for the average age to be 30?

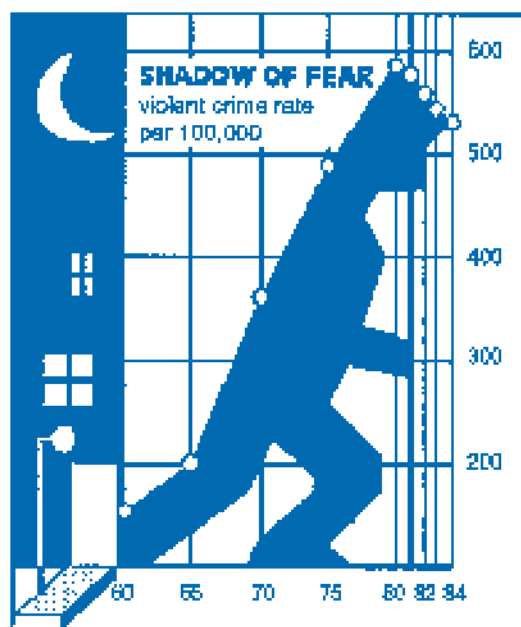
MATHEMATICS EXAMPLE 11: GROWING INCOMES?

Has the income of people in Zedland gone up or down in recent decades? The median money income per household fell: in 1970 it was 34 200 zed, in 1980 it was 30 500 zed and in 1990 31 200 zed. But the income per person increased: in 1970 13 500 zed, in 1980 13 850, and in 1990 15 777 zed.

A household consists of all people living together at the same address. Explain how it is possible for the household income to go down at the same time the per-person income has risen in Zedland.

MATHEMATICS EXAMPLE 12: RISING CRIMES

The following graph was taken from the weekly *Zedland News Magazine*:



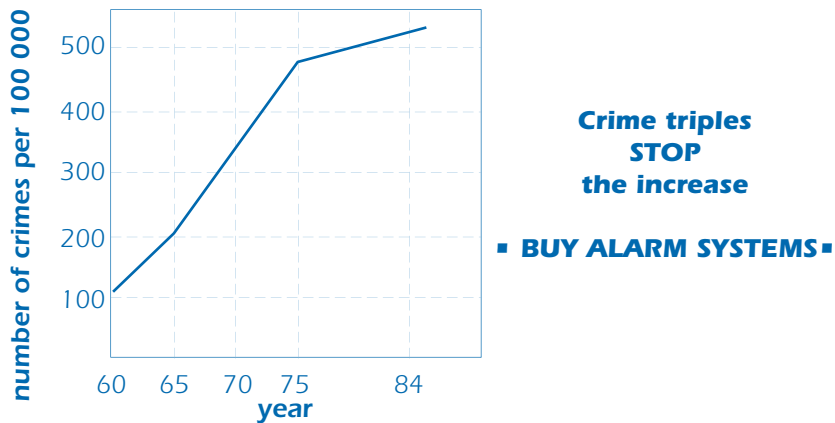
It shows the number of reported crimes per 100 000 inhabitants, starting with five-year intervals, then changing to one-year intervals.



Question 1: RISING CRIMES

How many reported crimes per 100 000 were there in 1960?

Manufacturers of alarm systems used the same data to produce the following graph:



Question 2: RISING CRIMES

How did the designers come up with this graph and why?

The police were not too happy with the graph from the alarm systems manufacturers because the police want to show how successful crime fighting has been.

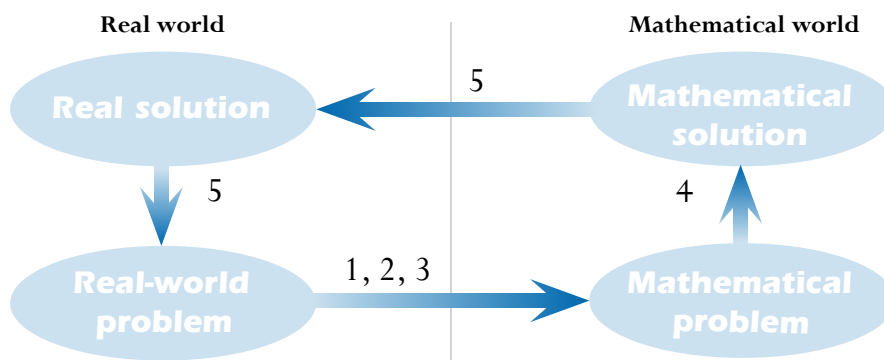
Design a graph to be used by the police to demonstrate that crime has decreased recently.

Mathematical processes

Mathematisation

PISA examines the ability of students to analyse, reason and communicate mathematical ideas effectively as they pose, formulate, solve and interpret mathematical problems in a variety of situations. Such problem solving requires students to use the mathematical processes, knowledge and skills they have acquired through schooling and life experiences. In PISA, the fundamental process that students use to solve real-life problems is referred to as mathematisation.

Figure 2.8 ■ The mathematisation cycle



The discussion above of the theoretical basis for the PISA mathematics framework outlines a five-step description of mathematisation. These steps are shown in Figure 2.8 and listed below:

1. Starting with a problem situated in reality.
2. Organising it according to mathematical concepts and identifying the relevant mathematics involved.
3. Gradually trimming away the reality through processes such as making assumptions, generalising and formalising. These processes promote the mathematical features of the situation and transform the real-world problem into a mathematical problem that faithfully represents the situation.
4. Solving the mathematical problem.
5. Making sense of the mathematical solution in terms of the real situation, including identifying the limitations of the solution.

Mathematisation first involves translating the problem from reality into mathematics. This process includes activities such as:

- Identifying the relevant mathematics with respect to a problem situated in reality.
- Representing the problem in a different way, including organising it according to mathematical concepts and making appropriate assumptions.
- Understanding the relationships between the language of the problem and the symbolic and formal language needed to understand it mathematically.
- Finding regularities, relations and patterns.
- Recognising aspects that are isomorphic with known problems.
- Translating the problem into mathematics *i.e.* to a mathematical model (de Lange, 1987).

As soon as a student has translated the problem into a mathematical form, the whole process can continue within mathematics. Students pose questions like: “Is there...?”, “If so, how many?”, “How do I find...?”, using known mathematical skills and concepts. They attempt to work on their model of the problem situation, adjust it, establish regularities, identify connections and create a good mathematical argument. This part of the mathematisation process is generally called the deductive part of the modelling cycle





(Blum, 1996; Schupp, 1988). However, processes other than strictly deductive ones may play a part in this stage as well. This part of the mathematisation process includes:

- Using and switching between different representations.
- Using symbolic, formal and technical language and operations.
- Refining and adjusting mathematical models, combining and integrating models.
- Argumentation.
- Generalisation.

The last step or steps in solving a problem involve reflecting on the whole mathematisation process and the results. Here students must interpret the results with a critical attitude and validate the whole process. Such reflection takes place at all stages of the process, but it is especially important at the concluding stage. Aspects of this reflecting and validating process are:

- Understanding the extent and limits of mathematical concepts.
- Reflecting on mathematical arguments and explaining and justifying results.
- Communicating the process and solution.
- Critiquing the model and its limits.

This stage is indicated in two places in Figure 2.8 by the label 5, where the mathematisation process passes from the mathematical solution to the real solution and is related back to the original real-world problem.

The cognitive mathematical competencies

The previous section focused on the major concepts and processes involved in mathematisation. An individual who is to engage successfully in mathematisation within a variety of situations, extra- and intra-mathematical contexts, and overarching ideas, needs to possess a number of mathematical competencies which, taken together, can be seen as constituting comprehensive mathematical competence. Each of these competencies can be possessed at different levels of mastery. Different parts of mathematisation draw differently upon these competencies, with regard to both the particular ones involved and the level of mastery required. PISA has decided to make use of eight characteristic cognitive mathematical competencies that rely, in their present form, on the work of Niss (1999) and his Danish colleagues. Similar formulations may be found in the work of many others (as indicated in Neubrand *et al.*, 2001). Some of the terms used, however, are used differently among different authors.

- *Thinking and reasoning*: this involves posing questions characteristic of mathematics (“Is there...?”, “If so, how many?”, “How do I find...?”); knowing the kinds of answers mathematics can offer to such questions; distinguishing between different kinds of statements (definitions, theorems, conjectures, hypotheses, examples, conditioned assertions); and understanding and handling the extent and limits of given mathematical concepts.
- *Argumentation*: this involves knowing what mathematical proofs are and how they differ from other kinds of mathematical reasoning; following and assessing chains of mathematical arguments of different types; possessing a feel for heuristics (“What can or cannot happen, and why?”); and creating and expressing mathematical arguments.
- *Communication*: this involves expressing oneself, in a variety of ways, on matters with a mathematical content, in oral as well as in written form, and understanding others’ written or oral statements about such matters.
- *Modelling*: this involves structuring the field or situation to be modelled; translating reality into mathematical structures; interpreting mathematical models in terms of reality; working with a mathematical model; validating the model; reflecting, analysing and offering a critique of a model and its results; communicating about the model and its results (including the limitations of such results); and monitoring and controlling the modelling process.



- *Problem posing and solving*: this involves posing, formulating and defining different kinds of mathematical problems (for example “pure”, “applied”, “open ended” and “closed”), and solving different kinds of mathematical problems in a variety of ways.
- *Representation*: this involves decoding and encoding, translating, interpreting and distinguishing between different forms of representation of mathematical objects and situations; the interrelationships between the various representations; and choosing and switching between different forms of representation, according to situation and purpose.
- *Using symbolic, formal and technical language and operations*: this involves decoding and interpreting symbolic and formal language, and understanding its relationship to natural language; translating from natural language to symbolic/formal language; handling statements and expressions containing symbols and formulae; and using variables, solving equations and undertaking calculations.
- *Use of aids and tools*: this involves knowing about and being able to make use of various aids and tools (including information technology tools) that may assist mathematical activity and knowing about the limitations of such aids and tools.

PISA does not test these cognitive mathematical competencies individually. There is considerable overlap among them, and when using mathematics, it is usually necessary to draw simultaneously on many of them, so that any effort to assess individual ones is likely to result in artificial tasks and unnecessary compartmentalisation. The particular mathematical processes, knowledge and skills students are able to display vary considerably among individuals. This is partially because all learning occurs through experiences, “with individual knowledge construction occurring through the processes of interaction, negotiation, and collaboration (de Corte, Greer and Verschaffel, 1996). PISA assumes that much of students’ mathematics is learned in schools and acknowledges that an understanding of the domain is acquired gradually. More formal and abstract ways of representing and reasoning emerge over time as a consequence of engagement in activities designed to help informal ideas evolve. Mathematical understanding is also acquired through experience involving interactions in a variety of social situations or contexts.

Some structure is needed in order to productively describe and report students’ capabilities, as well as their strengths and weaknesses from an international perspective. One way of providing this in a comprehensible and manageable way is to describe clusters of cognitive mathematical competencies, based on the kinds of cognitive demands needed to solve different mathematical problems.

Competency clusters

PISA has chosen to describe the cognitive activities that these cognitive mathematical competencies encompass according to three clusters: the *reproduction* cluster, the *connections* cluster and the *reflection* cluster. In the following sections the three clusters are described and the ways in which the individual competencies are played out in each cluster are discussed.

The reproduction cluster

The mathematical competencies in this cluster involve the reproduction of practised knowledge. They include those mathematical processes, knowledge and skills most commonly targeted on standardised assessments and classroom tests. These are knowledge of facts and of common problem representations, recognition of equivalents, recollection of familiar mathematical objects and properties, performance of routine procedures, application of standard algorithms and technical skills, manipulation of expressions containing symbols and formulae in standard form, and carrying out computations.

- *Thinking and reasoning*: this involves posing the most basic forms of questions (“How many...?”, “How much is...?”) and understanding the corresponding kinds of answers (“so many...”, “this much...”); distinguishing between definitions and assertions; understanding and handling mathematical concepts in the sorts of contexts in which they were first introduced or have subsequently been practised.
- *Argumentation*: this involves following and justifying standard quantitative processes, including computational processes, statements and results.



- **Communication:** this involves understanding and expressing oneself orally and in writing about simple mathematical matters, such as reproducing the names and the basic properties of familiar objects, citing computations and their results, usually not in more than one way.
- **Modelling:** this involves recognising, recollecting, activating and exploiting well-structured, familiar models; interpreting back and forth between such models (and their results) and reality; and elementary communication about model results.
- **Problem posing and solving:** this involves posing and formulating problems by recognising and reproducing practised standard pure and applied problems in closed form; and solving such problems by invoking and using standard approaches and procedures, typically in one way only.
- **Representation:** this involves decoding, encoding and interpreting familiar, practised standard representations of well known mathematical objects. Switching between representations is involved only when the switching itself is an established part of the representations implied.
- **Using symbolic, formal and technical language and operations:** this involves decoding and interpreting routine basic symbolic and formal language practised in well known contexts and situations; and handling simple statements and expressions containing symbols and formulae, including using variables, solving equations and undertaking calculations by routine procedures.
- **Use of aids and tools:** this involves knowing about and being able to use familiar aids and tools in contexts, situations and ways close to those in which their use was introduced and practised.

Assessment items measuring the *reproduction* cluster can be described with the following key descriptors: reproducing practised material and performing routine operations.

The following are examples of *reproduction* cluster items that could be used in the assessment:

MATHEMATICS EXAMPLE 13

Solve the equation $7x-3 = 13x+15$

MATHEMATICS EXAMPLE 14

What is the average of 7, 12, 8, 14, 15, 9?

MATHEMATICS EXAMPLE 15

1000 zed is put in a savings account at a bank, with an interest rate of 4%. How many zed will there be in the account after one year?

REACTION TIME, is an example of a PISA item used in the 2003 Field Trial. EXPORTS was used in the 2003 main survey.

MATHEMATICS EXAMPLE 16: REACTION TIME



In a sprinting event, the reaction time is the time interval between the starter's gun firing and the athlete leaving the starting block. The final time includes both this reaction time, and the running time.

The following table gives the reaction time and the final time of 8 runners in a 100 metre sprint race.

Lane	Reaction time (sec)	Final time (sec)
1	0.147	10.09
2	0.136	9.99
3	0.197	9.87
4	0.180	Did not finish the race
5	0.210	10.17
6	0.216	10.04
7	0.174	10.08
8	0.193	10.13

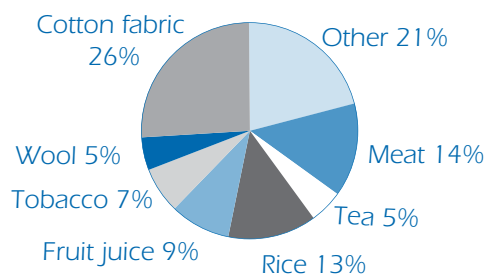
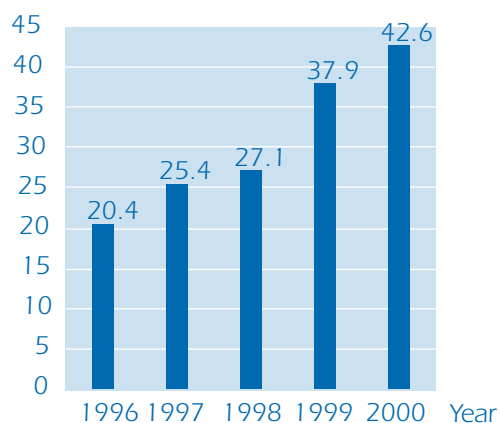
Question 1: REACTION TIME

Identify the Gold, Silver and Bronze medallists from this race. Fill in the table below with the medallists' lane number, reaction time and final time.

Medal	Lane	Reaction time (sec)	Final time (sec)
GOLD			
SILVER			
BRONZE			

MATHEMATICS EXAMPLE 17: EXPORTS

The graphics show information about exports from Zedland, a country that uses zeds as its currency.



Question 1: EXPORTS

What was the value of fruit juice exported from Zedland in 2000?

- A. 1.8 million zeds.
- B. 2.3 million zeds.
- C. 2.4 million zeds.
- D. 3.4 million zeds.
- E. 3.8 million zeds.



In order to clarify the boundary for items from the *reproduction* cluster, the *SAVINGS ACCOUNT* problem described in Example 3 provides an example that does not belong to the *reproduction* cluster. This problem takes most students beyond the simple application of a routine procedure and requires the application of a chain of reasoning and sequence of computational steps that are not characteristic of mathematical processes, knowledge and skills in the *reproduction* cluster.

The connections cluster

The *connections* cluster builds on the *reproduction* cluster by applying problem solving to situations that are not routine but still involve familiar or quasi-familiar settings. These mathematical processes knowledge and skills include the following:

- *Thinking and reasoning*: this involves posing questions (“How do I find...?”, “Which mathematics is involved...?”) and understanding the corresponding kinds of answers (provided by means of tables, graphs, algebra, figures, etc.); distinguishing between definitions and assertions and between different kinds of assertions; and understanding and handling mathematical concepts in contexts that are slightly different from those in which they were first introduced or have subsequently been practised.
- *Argumentation*: this involves simple mathematical reasoning without distinguishing between proofs and broader forms of argument and reasoning; following and assessing chains of mathematical arguments of different types; and possessing a feel for heuristics (e.g. “What can or cannot happen, or be the case, and why?”, “What do I know and what do I want to obtain?”).
- *Communication*: this involves understanding and expressing oneself orally and in writing about mathematical matters ranging from reproducing the names and basic properties of familiar objects and explaining computations and their results (usually in more than one way), to explaining matters that include relationships. It also involves understanding others’ written or oral statements about such matters.
- *Modelling*: this involves structuring the field or situation to be modelled; translating reality into mathematical structures in contexts that are not too complex but nevertheless different from what students are usually familiar with. It also involves interpreting back and forth between models (and their results) and reality, including aspects of communication about model results.
- *Problem posing and solving*: this involves posing and formulating problems beyond the reproduction of practised standard pure and applied problems in closed form and solving such problems by invoking and using standard approaches and procedures, as well as more independent problem-solving processes in which connections are made between different mathematical areas and modes of representation and communication (schemata, tables, graphs, words, pictures).
- *Representation*: this involves decoding, encoding and interpreting familiar and less familiar representations of mathematical objects; choosing and switching between different forms of representation of mathematical objects and situations; and translating and distinguishing between different forms of representation.
- *Using symbolic, formal and technical language and operations*: This involves decoding and interpreting basic symbolic and formal language in less well-known contexts and situations, and handling statements and expressions containing symbols and formulae, including using variables, solving equations and undertaking calculations using familiar procedures.
- *Use of aids and tools*: this involves knowing about and using familiar aids and tools in contexts, situations and ways that are different from those in which their use was introduced and practised.

Items associated with this cluster usually require some evidence of the integration and connection of material from the various overarching ideas, or from different mathematical curriculum strands, or the linking of different representations of a problem.

Assessment items measuring the *connections* cluster might be described with the following key descriptors: integrating, connecting and modest extension of practised material.

Examples of connections cluster items

One example of a *connections* cluster item was given in the *SAVINGS ACCOUNT* problem described in Example 3. Other examples of *connections* cluster items follow.



MATHEMATICS EXAMPLE 18: DISTANCE

Mary lives two kilometres from school, Martin five.

Question 1: DISTANCE

How far do Mary and Martin live from each other?

When this problem was originally presented to teachers, many of them rejected it on the grounds that it was too easy – one could easily see that the answer was three. Another group of teachers argued that it was not a good item because there was no answer – meaning there was not one single numerical answer. A third reaction was that it was not a good item because there were many possible answers since without further information, the most that could be concluded was that Mary and Martin live somewhere between three and seven kilometres from one another, and that was not desirable for an item. A small group thought it was an excellent item because one must understand the question, it is real problem solving because there is no strategy known to the student, and it is beautiful mathematics, although you have no clue how students will solve the problem. It is this last interpretation that associates the problem with the mathematical processes, knowledge and skills in the *connections* cluster.

MATHEMATICS EXAMPLE 19: THE OFFICE RENTING

The following two advertisements appeared in a daily newspaper in a country where the units of currency are zeds.

BUILDING A	BUILDING B
Office space available	Office space available
58-95 square metres	35-260 square metres
475 zeds per month	90 zeds per square metre
100-120 square metres	per year
800 zeds per month	

Question 1: THE OFFICE RENTING

If a company is interested in renting an office of 110 square metres in that country for a year, at which office building, A or B, should the company rent the office in order to get the lower price? Show your work. [© IEA/TIMSS]

MATHEMATICS EXAMPLE 20: THE PIZZA

A pizzeria serves two round pizzas of the same thickness in different sizes. The smaller one has a diameter of 30 cm and costs 30 zeds. The larger one has a diameter of 40 cm and costs 40 zeds. [© PRIM, Stockholm Institute of Education]

Question 1: THE PIZZA

Which pizza is better value for money? Show your reasoning.

In both of these problems, students are required to translate a real-life situation into mathematical language, to develop a mathematical model that enables them to make a suitable comparison, to check that the solution fits in with the initial question context and to communicate the result. These are all activities associated with the *connections* cluster.



The reflection cluster

The mathematical processes, knowledge and skills in this cluster include an element of reflectiveness on the part of the student about the processes needed or used to solve a problem. They relate to students' abilities to plan solution strategies and implement them in problem settings that contain more elements and may be more 'original' (or unfamiliar) than those in the *connections* cluster. In addition to the processes, knowledge and skills described for the *connections* cluster, the *reflection* cluster includes the following:

- **Thinking and reasoning:** this involves posing questions (“How do I find...?”, “Which mathematics are involved...?”, “What are the essential aspects of the problem or situation...?”) and understanding the corresponding kinds of answers (provided by tables, graphs, algebra, figures, specification of key points, etc.); distinguishing between definitions, theorems, conjectures, hypotheses and assertions about special cases, and reflecting upon or actively articulating these distinctions; understanding and handling mathematical concepts in contexts that are new or complex; and understanding and handling the extent and limits of given mathematical concepts, and generalising results.
- **Argumentation:** this involves simple mathematical reasoning, including distinguishing between proving and proofs and broader forms of argument and reasoning; following, assessing and constructing chains of mathematical arguments of different types; and using heuristics (e.g. “What can or cannot happen, or be the case, and why?”, “What do I know, and what do I want to obtain?”, “Which properties are essential?”, “How are the objects related?”).
- **Communication:** this involves understanding and expressing oneself orally and in writing about mathematical matters ranging from reproducing the names and basic properties of familiar objects, and explaining computations and their results (usually in more than one way), to explaining matters that include complex relationships, including logical relationships. It also involves understanding others' written or oral statements about such matters.
- **Modelling:** this involves structuring the field or situation to be modelled; translating reality into mathematical structures in contexts that may be complex or largely different from what students are usually familiar with; interpreting back and forth between models (and their results) and reality, including aspects of communication about model results: gathering information and data, monitoring the modelling process and validating the resulting model. It also includes reflecting through analysing, offering a critique, and engaging in more complex communication about models and modelling.
- **Problem posing and solving:** this involves posing and formulating problems well beyond the reproduction of practised standard pure and applied problems in closed form; solving such problems by invoking and using standard approaches and procedures, but also more original problem-solving processes in which connections are being made between different mathematical areas and modes of representation and communication (schemata, tables, graphs, words, pictures). It also involves reflecting on strategies and solutions.
- **Representation:** this involves decoding, encoding and interpreting familiar and less familiar representations of mathematical objects; choosing and switching between different forms of representation of mathematical objects and situations, and translating and distinguishing between different forms of representation. It further involves the creative combination of representations and the invention of non-standard ones.
- **Using symbolic, formal and technical language and operations:** this involves decoding and interpreting symbolic and formal language practised in unknown contexts and situations, and handling statements and expressions containing symbols and formulae, including using variables, solving equations and undertaking calculations. It also involves the ability to deal with complex statements and expressions and with unfamiliar symbolic or formal language, and to understand and to translate between such language and natural language.
- **Use of aids and tools:** this involves knowing about and using familiar or unfamiliar aids and tools in contexts, situations and ways that are quite different from those in which their use was introduced and practised. It also involves knowing about limitations of aids and tools.

Assessment items measuring the *reflection* cluster might be described with the following key descriptors: advanced reasoning, argumentation, abstraction, generalisation and modelling applied to new contexts.



Examples of reflection cluster items

MATHEMATICS EXAMPLE 21: STUDENT HEIGHTS

In a mathematics class one day, the heights of all students were measured. The average height of boys was 160 cm, and the average height of girls was 150 cm. Alena was the tallest – her height was 180 cm. Zdenek was the shortest – his height was 130 cm.

Two students were absent from class that day, but they were in class the next day. Their heights were measured, and the averages were recalculated. Amazingly, the average height of the girls and the average height of the boys did not change.

Question 1: STUDENT HEIGHTS

Which of the following conclusions can be drawn from this information?

Circle 'Yes' or 'No' for each conclusion.

Conclusion	Can this conclusion be drawn?
Both students are girls.	Yes / No
One of the students is a boy and the other is a girl.	Yes / No
Both students have the same height.	Yes / No
The average height of all students did not change.	Yes / No
Zdenek is still the shortest.	Yes / No

This problem is quite complicated in several ways. It requires very precise reading, as superficial reading will likely lead to misinterpretation. Norris and Phillips (2003) have argued that reading literacy is fundamental for science literacy. In a similar way, mathematical literacy also depends to some degree on reading skills. The item above demonstrates the importance of the receptive aspect of the communication competency, which is demanded strongly in this case. It also demonstrates how this aspect of *mathematical literacy* intersects with other types of literacy, specifically *reading literacy*. Furthermore, the wording of the problem makes it difficult to locate the crucial mathematical information. The intersection of other domains with the PISA definition and assessment of mathematics cannot be avoided; however, at the core of each assessment task there should be aspects that relate unambiguously to mathematical knowledge and skills. In this case, interpreting written information that contains mathematical data, and converting this information to a useful mathematical formulation are key mathematical challenges to be negotiated before a solution can be found.

The situation varies within the *class* and over *time*. The entity *class* is used while discussing the average for boys and for girls independently, but subsequently it is stated that Alena is the tallest (girl or student) and Zdenek the shortest (boy or student). If the students do not read carefully they will miss the fact that Alena is a girl and Zdenek a boy.

One obvious difficulty is the fact that the students have to combine the information from the first part of the stimulus (about the different heights) with the second part where the information about two missing students is presented. Here variation over *time* is seen: there are two students who were not present in the original setting, but have to be taken into account the next moment in time, so the entity *class* changes. However, the student solving the problem does not know whether the missing students are boys, girls or one of each. In addition, there is not one problem to solve, but five.

Furthermore, to be able to answer correctly the students need to understand mathematically the statistical concepts involved. The problem involves the ability to pose questions ("How do I know...?", "How do I find...?", "What are the possibilities...?" and "What happens if I...?") and the ability to understand and handle the concept of an average in texts that are complex, although the context is familiar.

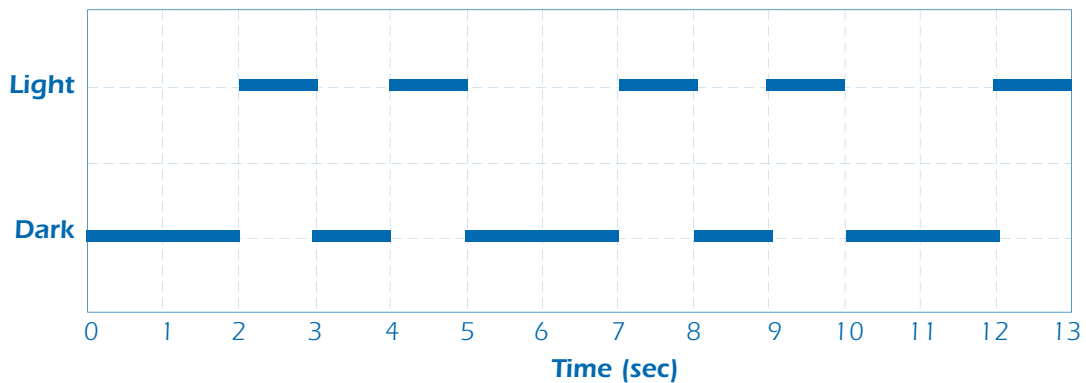
From this description it is clear that this item is not only challenging for students (as shown by the PISA results) but also clearly belongs to the *reflection* cluster.

MATHEMATICS EXAMPLE 22: LIGHTHOUSE

Lighthouses are towers with a light beacon on top. Lighthouses assist sea ships in finding their way at night when they are sailing close to the shore.

A lighthouse beacon sends out light flashes with a regular fixed pattern. Every lighthouse has its own pattern.

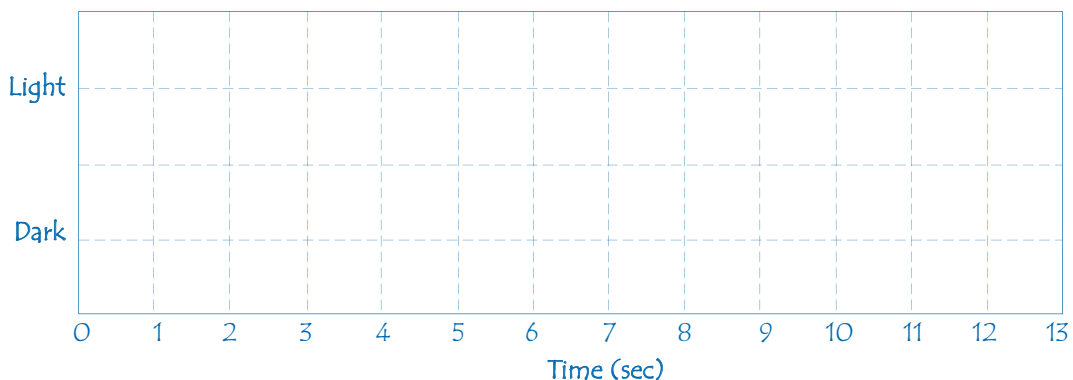
In the diagram below you see the pattern of a certain lighthouse. The light flashes alternate with dark periods.



It is a regular pattern. After some time the pattern repeats itself. The time taken by one complete cycle of a pattern, before it starts to repeat, is called the period. When you find the period of a pattern, it is easy to extend the diagram for the next seconds or minutes or even hours.

Question 1: LIGHTHOUSE

In the diagram below, make a graph of a possible pattern of light flashes of a lighthouse that sends out light flashes for 30 seconds per minute. The period of this pattern must be equal to 6 seconds.



In this example, the students must first understand the introduction in the sense that this kind of graphs is most likely unknown to them, as is the idea of periodicity. In addition, the question posed is of a very open nature: the students are asked to design a possible pattern of light flashes. Many students do not encounter this kind of constructive question at school. However, this constructive aspect is an essential component of being mathematically literate: using mathematical competencies not only in a passive or derived way, but in constructing an answer. Solving the problem demands satisfying two conditions: equal amounts of time light and dark ("30 seconds per minute"), and a period of six seconds. This combination makes it essential for the students to engage with periodicity at the conceptual level – this involves the *reflection* cluster.

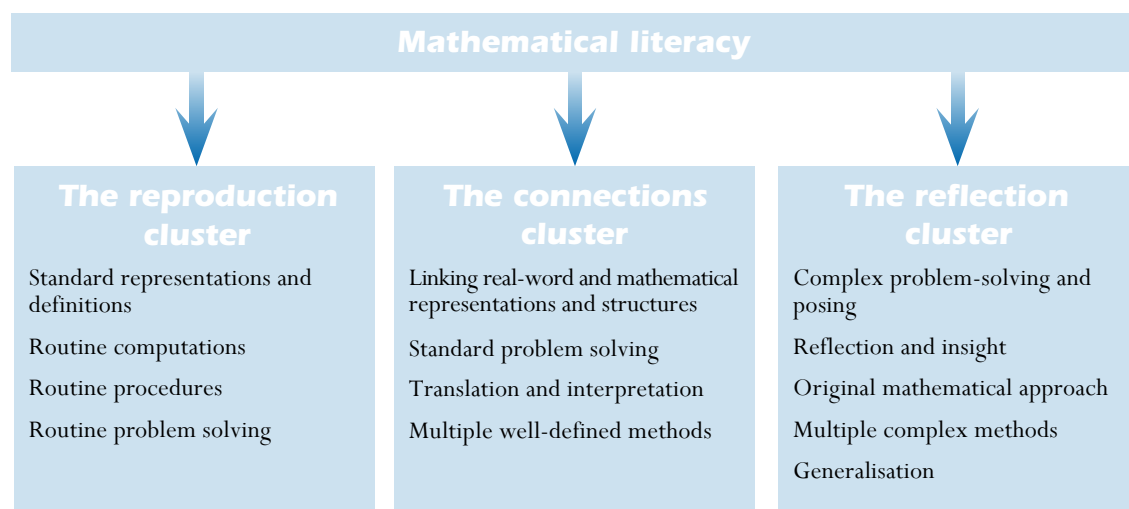


In this particular example, the context could be said to favour students living close to an ocean. It should be pointed out, however, that the PISA assessment of mathematics requires the capacity to use mathematics in contexts different from the local one. This ability to transfer is an essential competency. While certain students can be in a somewhat favourable position in certain contexts, and others in other contexts, the item by country analysis gives no indication that this is the case: landlocked countries did not perform differently from countries bordering on oceans.

Classification of items by competency cluster

Figure 2.9 summarises the distinctions between the clusters.

Figure 2.9 ■ Diagram representing the competency clusters



It is possible to use the descriptions outlined previously to classify mathematics items and thereby to assign them to one of the competency clusters. One way to do this is to analyse the demands of the item, then to rate each of the eight cognitive mathematical competencies for that item, according to which of the three clusters provide the most fitting description of item demands in relation to that competency. If any of the competencies were rated as fitting the description for the *reflection* cluster, then the item would be assigned to the *reflection* cluster. If not, but one or more of the competencies were rated as fitting the description for the *connections* cluster, then the item would be assigned to that cluster. Otherwise, the item would be assigned to the *reproduction* cluster, since all cognitive mathematical competencies would have been rated as fitting the descriptions for that cluster.



ASSESSING MATHEMATICS IN PISA

Task characteristics

This section considers, in more detail, features of the assessment tasks that are used to assess students. The nature of the tasks and the item format types are described below.

The nature of PISA mathematics tasks

PISA is an international survey of the knowledge and skills of 15-year-old students. All test items used should be suitable for the population of 15-year-old students in OECD countries.

Trained markers have access to items including some stimulus material or information, an introduction, the actual question and the required solution. In addition, for items with responses that cannot be automatically coded, a detailed coding scheme is developed to enable trained markers across the range of participating countries to code the student responses in a consistent and reliable way.

In an earlier section of this framework, the situations to be used for PISA mathematics items were discussed in some detail. For PISA, each item is set in one of four situation types: personal, educational/occupational, public and scientific. The items selected for the PISA mathematics instruments represent a spread across these situation types.

In addition, item contexts that can be regarded as authentic are preferred. That is, PISA values most highly the tasks that could be encountered in real-world situations and have a context that demands the use of mathematics to solve the problem in an authentic manner. Problems with extra-mathematical contexts that influence the solution and its interpretation are preferred as vehicles for assessing mathematical processes, knowledge and skills.

The selection of mathematics test items aims to ensure that the four overarching ideas are well represented. This framework requires the same of its mathematics test items. Items should relate predominantly to the overarching ideas (the phenomenological problem categories) described in the framework. Additionally, they should embody one or more of the mathematical processes that are described in the framework, and should be identified predominantly with one of the competency clusters.

The level of reading required to successfully engage with an item is considered very carefully when items are developed and selected for inclusion in the PISA test instrument. The wording of items is as simple and direct as possible. Care is also taken to avoid question contexts that would create a cultural bias.

Items selected for inclusion in the PISA test instruments represent a broad range of difficulties, to match the expected wide ability range of students participating in the PISA assessment. In addition, the major classifications of the framework (particularly competency clusters and overarching ideas) should, as much as possible, be represented by items of a wide range of difficulties. Item difficulties are established in an extensive field trial of test items prior to item selection for the main PISA survey.

Item types

When assessment instruments are devised, the impact of the item format type on student performance, and hence on the definition of the construct that is being assessed, must be carefully considered. This issue is particularly pertinent in a project such as PISA, where the large-scale cross-national context for testing puts serious constraints on the range of feasible item format types.

The PISA mathematics assessment employs a combination of items with open constructed-response types, closed constructed-response types and multiple-choice types. About equal numbers of each of these item format types are used in constructing the test instruments.

Based on experience in developing and using test items for PISA 2000, the multiple-choice type is generally regarded as most suitable for assessing items that would be associated with the *reproduction* and *connections* clusters. For an example of this item type, see Example 23, which shows an item that would be associated with the *connections* cluster and with a limited number of defined response options. To solve this problem, students



must translate the problem into mathematical terms, devise a model to represent the periodic nature of the context described, and extend the pattern to match the result with one of the given options.

MATHEMATICS EXAMPLE 23: SEAL

A seal has to breathe even if it is asleep. Martin observed a seal for one hour. At the start of his observation the seal dived to the bottom of the sea and started to sleep. In 8 minutes it slowly floated to the surface and took a breath.

In 3 minutes it was back at the bottom of the sea again and the whole process started over in a very regular way.

Question 1: SEAL

After one hour the seal was:

- A. At the bottom
- B. On its way up
- C. Breathing
- D. On its way down

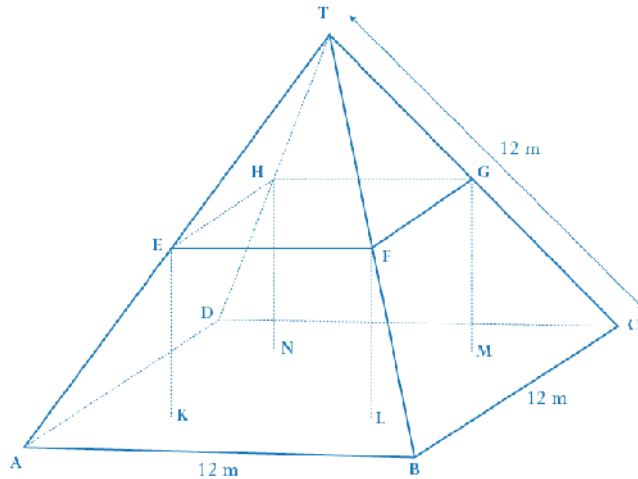
Other format types are often preferred for some of the higher-order goals and more complex processes. Closed constructed-response items can pose similar questions to multiple-choice items, but students are asked to produce a response that can be easily judged as either correct or incorrect. For items in this type, guessing is not likely to be a concern, and the provision of distractors (which influence the construct that is being assessed) is not necessary. For example, the problem in Example 24 has one correct answer and many possible incorrect answers.

MATHEMATICS EXAMPLE 24: FARMS

Here you see a photograph of a farmhouse with a roof in the shape of a pyramid.



Below is a student's mathematical model of the farmhouse roof with measurements added.



The attic floor, ABCD in the model, is a square. The beams that support the roof are the edges of a block (rectangular prism) EFGHKL MN. E is the middle of AT, F is the middle of BT, G is the middle of CT and H is the middle of DT. All the edges of the pyramid in the model have length 12 m.

Question 1: FARMS

Calculate the area of the attic floor ABCD.

The area of the attic floor ABCD = _____ m²

Open constructed-response items require a more extended response from the student, and the process of producing a response frequently involves higher-order cognitive activities. Often such items not only ask the student to produce a response, but also require the student to show their steps taken or explain how the answer was reached. The key feature of open constructed-response items is that students are allowed to demonstrate their abilities by providing solutions at a range of levels of mathematical complexity.

Example 25 was not used in any PISA assessment, it exemplifies the kinds of features that are present in open constructed-response items.

MATHEMATICS EXAMPLE 25: INDONESIA

Indonesia lies between Malaysia and Australia. Some data of the population of Indonesia and its distribution over the islands is shown in the following table:

Region	Surface area (km)	Percentage of total area	Population in 1980 (millions)	Percentage of total population
Java/Madura	132 187	6.95	91 281	61.87
Sumatra	473 606	24.86	27 981	18.99
Kalimantan (Borneo)	539 460	28.32	6 721	4.56
Sulawesi (Celebes)	189 216	9.93	10 377	7.04
Bali	5 561	0.30	2 470	1.68
Irian Jaya	421 981	22.16	1 145	5.02
TOTAL	1 905 569	100.00	147 384	100.00

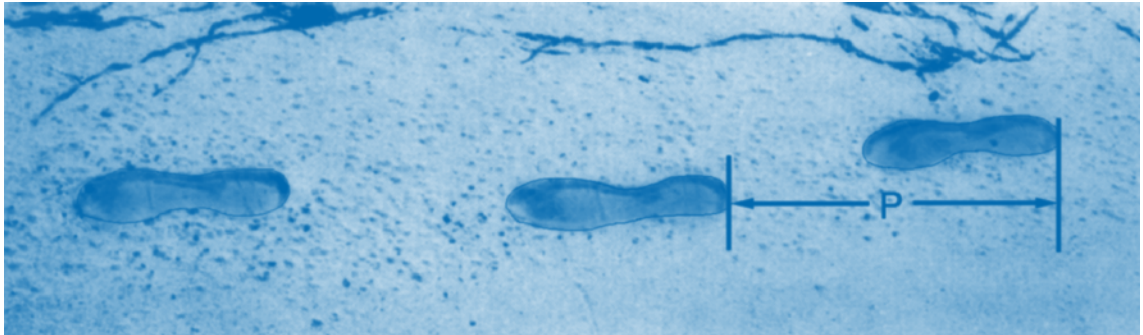
One of the main challenges for Indonesia is the uneven distribution of the population over the islands. From the table we can see that Java, which has less than 7% of the total area, has almost 62% of the population.

Source: de Lange and Verhage (1992). Used with permission.

Question 1: INDONESIA

Design a graph (or graphs) that shows the uneven distribution of the Indonesian population.

The following item is an example of an open constructed-response item that was used in PISA 2003:

MATHEMATICS EXAMPLE 26: WALKING

The picture shows the footprints of a man walking. The pacerlength P is the distance between the rear of two consecutive footprints.

For men, the formula, $\frac{n}{P} = 140$, gives an approximate relationship between n and P where,

n = number of steps per minute, and

P = pace length in metres.

Question 1: WALKING

Bernard knows his pacerlength is 0.80 metres. The formula applies to Bernard's walking.

Calculate Bernard's walking speed in metres per minute and in kilometres per hour. Show your working out.

About one-third of the mathematics items for PISA are open constructed-response items. The responses to these items require coding by trained individuals who implement a coding rubric that may require an element of professional judgement. Because of the potential for disagreement between coders of responses to these items, PISA implements coder reliability studies to monitor the extent of disagreement. Experience in these types of studies shows that clear coding rubrics can be developed and reliable scores can be obtained.

PISA makes some use of a unit format in which several items are linked to common stimulus material. Tasks of this format give students the opportunity to become involved with a context or problem by asking a series of questions of increasing complexity. The first few questions are typically multiple-choice or closed constructed-response items, while subsequent ones are typically open constructed-response items. This format can be used to assess each of the three clusters.

One reason for the use of common stimulus task formats is that they allow realistic tasks to be devised and the complexity of real-life situations to be reflected in them. Another reason relates to the efficient use of testing time, cutting down on the time required for a student to understand the problem. The need to make each scored point independent of others within a task is recognised and taken into account in the design of PISA tasks and the response coding and scoring rubrics. The importance of minimising bias that may result from using fewer situations is also recognised.

Assessment structure

In PISA 2003, when mathematics was the major PISA assessment domain, the test instruments contained a total of 210 minutes of testing time. The selected test items were arranged in seven clusters of items, with each item cluster representing 30 minutes of testing time, and the item clusters were placed in test booklets according to





a rotated test design. For the 2006 test cycle, when science was the major PISA assessment domain, less time was devoted to testing of mathematics, but the item clusters allocated to mathematics were constructed and rotated in a similar way. Similar arrangements apply to the mathematics test material for the PISA 2009 survey. In addition for 2009, clusters have been selected intact from those used in the 2006 survey to minimise any changed measurement effects arising from item placement factors.

The total testing time for mathematics is distributed as evenly as possible across the four overarching ideas (*space and shape, change and relationships, quantity and uncertainty*) and the four situations described in the framework (*personal, educational/occupational, public and scientific*). The proportion of items reflecting the three clusters (reproduction, connections and reflection) is about 1:2:1, respectively. Multiple-choice response types, closed constructed-response types, and open constructed-response types each represent about one-third of the items.

Aids and tools

PISA policy allows students to use calculators and other tools as they are normally used in school.

This represents the most authentic assessment of what students can achieve, and provides the most informative comparison of the performance of education systems. A system's choice to allow students to access and use calculators is no different, in principle, from other instructional policy decisions that are made by systems and are not controlled for by PISA.

Students who are used to having a calculator available to assist them in answering questions would be disadvantaged if this resource were taken away.

REPORTING PROFICIENCY IN MATHEMATICS

To summarise data from responses to the PISA test instruments, a six-level described performance scale was created (Masters and Forster, 1996; Masters, Adams, and Wilson, 1999). Statistically created, the scale uses an item response modelling approach to scale ordered outcome data. The overall scale is used to describe the nature of performance by classifying the student performances of different countries in terms of the five described performance levels and thus providing a frame of reference for international comparisons.

For the reporting of results from PISA 2003 when mathematics was the major domain, consideration was given to the development of a number of separate reporting scales. Such subscales could most obviously be based on the three clusters or the four overarching ideas. Decisions about the development of separate reporting scales were made on a variety of grounds, including psychometric considerations, following analysis of the data generated by the PISA assessments. To facilitate these possibilities, it was necessary to ensure that sufficient items were selected for inclusion in the PISA test instrument from each potential reporting category. Moreover, items within each such category needed to have a suitably wide range of difficulties. The balance of items across these categories has been broadly maintained in subsequent PISA administrations, but student outcomes have not been reported according to the content-based subscales for mathematics when it was a minor domain, such as in 2006 and 2009.

The competency clusters described earlier in this framework reflect conceptual categories of broadly increasing cognitive demand and complexity, but do not strictly reflect a hierarchy of student performances based on item difficulty. Conceptual complexity is only one component of item difficulty that influences levels of performance. Others include familiarity, recent opportunity to learn and practice, etc. Thus, a multiple-choice item involving the *reproduction* cluster (for example, "Which of the following is a rectangular parallelepiped?" followed by pictures of a ball, a can, a box, and a square) may be very easy for students who have been taught the meaning of these terms, but very difficult for others because of their lack of familiarity with the terminology used. While it is possible to imagine relatively difficult *reproduction* cluster items and relatively easy *reflection* cluster items, one would expect a broadly positive relationship between competency clusters and item difficulty.



Factors that underpin increasing levels of item difficulty and mathematical proficiency include the following:

- *The kind and degree of interpretation and reflection required:* this includes the nature of demands arising from the problem context; the extent to which the mathematical demands of the problem are apparent or to which students must impose their own mathematical construction on the problem, and the extent to which insight, complex reasoning and generalisation are required.
- *The kind of representation skills required:* these include problems where only one mode of representation is used, as well as problems where students have to switch between different modes of representation or search for appropriate modes of representation themselves.
- *The kind and level of mathematical skill required:* these include single-step problems requiring students to reproduce basic mathematical facts and perform simple computation processes, multi-step problems involving more advanced mathematical knowledge, complex decision-making, information processing, and problem solving and modelling skills.
- *The kind and degree of mathematical argumentation required:* these include problems where no arguing is necessary, problems where students may apply well-known arguments, and problems where students have to create mathematical arguments, understand other people's argumentation or judge the correctness of given arguments or proofs.

At the lowest described proficiency level, students typically carry out single-step processes that involve the recognition of familiar contexts and mathematically well-formulated problems, reproducing well-known mathematical facts or processes, and applying simple computational skills.

At higher proficiency levels, students typically carry out more complex tasks involving more than a single processing step. They also combine different pieces of information or interpret different representations of mathematical concepts or information, recognising which elements are relevant and important and how they relate to one another. They typically work with given mathematical models or formulations, frequently in algebraic form, to identify solutions, or they carry out a small sequence of processing or calculation steps to produce a solution.

At the highest proficiency level, students take a more creative and active role in their approach to mathematical problems. They typically interpret more complex information and negotiate a number of processing steps. They produce a formulation of a problem and often develop a suitable model that facilitates its solution. Students at this level can typically identify and apply relevant tools and knowledge in an unfamiliar problem context. They likewise demonstrate insight in identifying a suitable solution strategy, and display other higher-order cognitive processes such as generalisation, reasoning and argumentation to explain or communicate results.

In Figure 2.10 the six described levels of mathematical proficiency developed for the PISA 2003 survey administration are presented, together with their associated scores on the PISA mathematics scale. The same descriptions were used in reporting the 2006 survey outcomes, and will also be used to report the PISA 2009 outcomes.

Figure 2.10 ■ Summary descriptions of the six proficiency levels in mathematics

Lower score limit	What students can typically do at each level
Level 6 669.3	At Level 6 students can conceptualise, generalise, and utilise information based on their investigations and modelling of complex problem situations. They can link different information sources and representations and flexibly translate among them. Students at this level are capable of advanced mathematical thinking and reasoning. These students can apply this insight and understandings along with a mastery of symbolic and formal mathematical operations and relationships to develop new approaches and strategies for attacking novel situations. Students at this level can formulate and precisely communicate their actions and reflections regarding their findings, interpretations, arguments, and the appropriateness of these to the original situations.
Level 5 607.0	At Level 5 students can develop and work with models for complex situations, identifying constraints and specifying assumptions. They can select, compare, and evaluate appropriate problem solving strategies for dealing with complex problems related to these models. Students at this level can work strategically using broad, well-developed thinking and reasoning skills, appropriate linked representations, symbolic and formal characterisations, and insight pertaining to these situations. They can reflect on their actions and formulate and communicate their interpretations and reasoning.
Level 4 544.7	At Level 4 students can work effectively with explicit models for complex concrete situations that may involve constraints or call for making assumptions. They can select and integrate different representations, including symbolic ones, linking them directly to aspects of real-world situations. Students at this level can utilise well-developed skills and reason flexibly, with some insight, in these contexts. They can construct and communicate explanations and arguments based on their interpretations, arguments, and actions.
Level 3 482.4	At Level 3 students can execute clearly described procedures, including those that require sequential decisions. They can select and apply simple problem solving strategies. Students at this level can interpret and use representations based on different information sources and reason directly from them. They can develop short communications reporting their interpretations, results and reasoning.
Level 2 420.1	At Level 2 students can interpret and recognise situations in contexts that require no more than direct inference. They can extract relevant information from a single source and make use of a single representational mode. Students at this level can employ basic algorithms, formulae, procedures, or conventions. They are capable of direct reasoning and making literal interpretations of the results.
Level 1 357.8	At Level 1 students can answer questions involving familiar contexts where all relevant information is present and the questions are clearly defined. They are able to identify information and to carry out routine procedures according to direct instructions in explicit situations. They can perform actions that are obvious and follow immediately from the given stimuli.

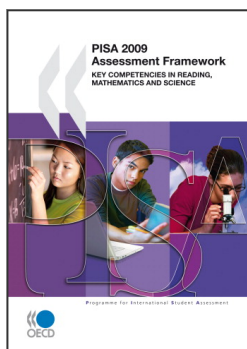
CONCLUSION

The aim of the PISA study, with regard to mathematics, is to develop indicators that show, from the perspective of the use of mathematics, how effectively countries have prepared their 15-year-old students to become active, reflective and intelligent citizens. To achieve this, PISA has developed assessments that focus on determining the extent to which students can use what they have learned. PISA emphasises mathematical processes, knowledge and understanding to solve problems that arise out of day-to-day experience, and provides a variety of problems with varying degrees of built-in guidance and structure, while simultaneously pushing towards authentic problems where students must do the thinking themselves.



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