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PISA 2012 Mathematics Framework

The PISA 2012 mathematics framework explains the theoretical underpinnings of the PISA mathematics assessment, including a new formal definition of mathematical literacy, the mathematical processes which students undertake when using mathematical literacy, and the fundamental mathematical capabilities which underlie those processes. The framework describes how mathematical content knowledge is organised into four content categories and outlines the content knowledge that is relevant to an assessment of 15-year-old students. It describes four categories of contexts in which students will face mathematical challenges. The framework specifies the proportions of items from each of the four content and context categories, each response format and each process, and describes the rotating booklet designs and questionnaires. Items of a range of difficulty are required. The optional computer-based assessment for mathematics is described, with discussion of the rationale and potential for future development. The categorisations are illustrated with seven units used in PISA surveys and field trials. Multiple quality control measures are described. The PISA assessment will measure how effectively countries are preparing students to use mathematics in every aspect of their personal, civic and professional lives, as part of their constructive, engaged and reflective citizenship.

INTRODUCTION

The assessment of mathematics has particular significance for PISA 2012, as mathematics is the major domain assessed. Although mathematics was assessed by PISA in 2000, 2003, 2006 and 2009, the domain was the main area of focus only in 2003.

The return of mathematics as the major domain in PISA 2012 provides the opportunity to make comparisons in student performance over time, but it also provides the opportunity to re-examine what is assessed in light of changes that have occurred in the field and in instructional policies and practices. An inherent challenge is developing an updated, state-of-the-art mathematics framework while retaining psychometric links to past mathematics assessments through a forward-looking and backward-looking trend line. The PISA 2012 framework is designed to make mathematics relevant to 15-year-old students more clear and explicit, while ensuring that the items developed remain set in meaningful and authentic contexts. The mathematical modelling cycle, used in earlier frameworks (e.g. OECD, 2003) to describe the stages individuals go through in solving contextualised problems, remains a key feature of the PISA 2012 framework. It is used to help define the mathematical processes in which students engage as they solve problems – processes that are being used for the first time in 2012 as a primary reporting dimension. A new optional computer-based assessment of mathematics (CBAM) is also available for countries in 2012.

The PISA 2012 mathematics framework is organised into several major sections. The first section, "Definition of mathematical literacy", explains the theoretical underpinnings of the PISA mathematics assessment, including the formal definition of the mathematical literacy construct. The second section, "Organising the domain", describes three aspects: *i*) the mathematical processes and the fundamental mathematical capabilities (in previous frameworks the "*competencies*") underlying those processes; *ii*) the way mathematical content knowledge is organised in the PISA 2012 framework, and the content knowledge that is relevant to an assessment of 15-year-old students (sub-scores are being reported for both the three mathematical process categories and the four mathematical content categories); *iii*) the contexts in which students will face mathematical challenges. The third section, "Assessing mathematical literacy", outlines structural issues about the assessment, including a test blueprint and other technical information. The several addenda include further descriptions of the fundamental mathematical capabilities, several illustrative PISA items and a reference list.

This framework was written under the guidance of the Mathematics Expert Group (MEG), a body appointed by the main PISA contractors with the approval of the PISA Governing Board (PGB). The ten MEG members include mathematicians, mathematics educators, and experts in assessment, technology, and education research from a range of countries. In addition, to secure more extensive input and review, a draft of the PISA 2012 mathematics framework was circulated for feedback to over 170 mathematics experts from over 40 countries. Achieve and the Australian Council for Educational Research (ACER), the two organisations contracted by the Organisation for Economic Co-operation and Development (OECD) to manage framework development, also conducted various research efforts to inform and support development work. Framework development and the PISA programme generally have been supported and informed by the ongoing work of participating countries (e.g. the research described in the 2010 OECD publication *Pathways to Success: How Knowledge and Skills at Age 15 Shape Future Lives in Canada*).

DEFINING MATHEMATICAL LITERACY

An understanding of mathematics is central to a young person's preparedness for life in modern society. A growing proportion of problems and situations encountered in daily life, including in professional contexts, require some level of understanding of mathematics, mathematical reasoning and mathematical tools, before they can be fully understood and addressed. Mathematics is a critical tool for young people as they confront issues and challenges in personal, occupational, societal, and scientific aspects of their lives. It is thus important to have an understanding of the degree to which young people emerging from school are adequately prepared to apply mathematics to understanding important issues and solving meaningful problems. An assessment at age 15 provides an early indication of how individuals may respond in later life to the diverse array of situations they will encounter that involve mathematics.

As the basis for an international assessment of 15-year-old students, it is reasonable to ask: "What is important for citizens to know and be able to do in situations that involve mathematics?" More specifically, what does competency in mathematics mean for a 15-year-old, who may be emerging from school or preparing to pursue more specialised training for a career or university admission? It is important that the construct of mathematical literacy, which is used in this report to denote the capacity of individuals to formulate, employ, and interpret mathematics in a variety of contexts, not be perceived as synonymous with minimal, or low-level, knowledge and skills. Rather, it is intended to describe

the capacities of individuals to reason mathematically and use mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena. This conception of mathematical literacy supports the importance of students developing a strong understanding of concepts of pure mathematics and the benefits of being engaged in explorations in the abstract world of mathematics. The construct of mathematical literacy, as defined for PISA, strongly emphasises the need to develop students' capacity to use mathematics in context, and it is important that they have rich experiences in their mathematics classrooms to accomplish this. This is true for those 15-year-old students who are close to the end of their formal mathematics training, as well as those who will continue with the formal study of mathematics. In addition, it can be argued that for almost all students, the motivation to learn mathematics increases when they see the relevance of what they are learning to the world outside the classroom and to other subjects.

Mathematical literacy naturally transcends age boundaries. However, its assessment for 15-year-olds must take into account relevant characteristics of these students; hence, there is a need to identify age-appropriate content, language and contexts. This framework distinguishes between broad categories of content that are important to mathematical literacy for individuals generally, and the specific content topics that are appropriate for 15-year-old students. Mathematical literacy is not an attribute that an individual either has or does not have. Rather, mathematical literacy is an attribute that is on a continuum, with some individuals being more mathematically literate than others – and with the potential for growth always present.

For the purposes of PISA 2012, mathematical literacy is defined as follows:

> *Mathematical literacy is an individual's capacity to formulate, employ, and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena. It assists individuals to recognise the role that mathematics plays in the world and to make the well-founded judgments and decisions needed by constructive, engaged and reflective citizens.*

Some explanatory remarks are provided below to highlight and clarify aspects of the definition that are particularly important.

A view of students as active problem solvers in PISA 2012

The focus of the language in the definition of mathematical literacy is on active engagement in mathematics, and is intended to encompass reasoning mathematically and using mathematical concepts, procedures, facts and tools in describing, explaining and predicting phenomena. In particular, the verbs 'formulate,' 'employ,' and 'interpret' point to the three processes in which students as active problem solvers will engage. *Formulating* mathematics involves identifying opportunities to apply and use mathematics – seeing that mathematics can be applied to understand or resolve a particular problem or challenge presented. It includes being able to take a situation as presented and transform it into a form amenable to mathematical treatment, providing mathematical structure and representations, identifying variables and making simplifying assumptions to help solve the problem or meet the challenge. *Employing* mathematics involves applying mathematical reasoning and using mathematical concepts, procedures, facts and tools to derive a mathematical solution. It includes performing calculations, manipulating algebraic expressions and equations or other mathematical models, analysing information in a mathematical manner from mathematical diagrams and graphs, developing mathematical descriptions and explanations and using mathematical tools to solve problems. *Interpreting* mathematics involves reflecting upon mathematical solutions or results and interpreting them in the context of a problem or challenge. It includes evaluating mathematical solutions or reasoning in relation to the context of the problem and determining whether the results are reasonable and make sense in the situation.

The language of the definition is also intended to integrate the notion of mathematical modelling, which has historically been a cornerstone of the PISA framework for mathematics (e.g. OECD, 2003), into the PISA 2012 definition of mathematical literacy. As individuals use mathematics and mathematical tools to solve problems in contexts, their work progresses through a series of stages. Figure 1.1 shows an overview of the major constructs of this framework and indicates how they relate to each other.

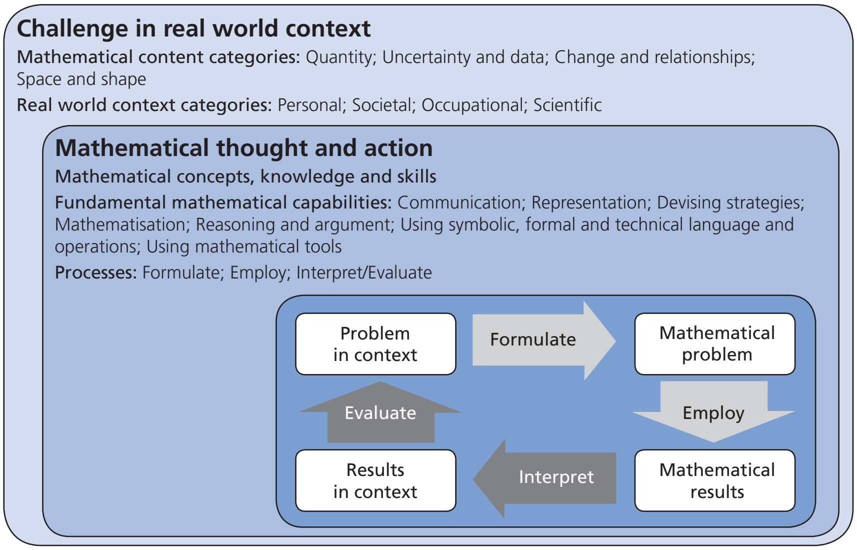
- The outer-most box in Figure 1.1 shows that mathematical literacy takes place in the context of a challenge or problem that arises in the real world. In this framework, these challenges are characterised in two ways. The context categories, which will be described in detail later in this document, identify the areas of life from which the problem arises. The context may be of a *personal* nature, involving problems or challenges that might confront an individual or one's family or peer group. The problem might instead be set in a *societal* context (focusing on one's community – whether it be local, national, or global), an *occupational* context (centred on the world of work), or a *scientific* context (relating

to the application of mathematics to the natural and technological world). A problem is also characterised by the nature of the mathematical phenomenon that underlies the challenge. The four mathematical content categories identify broad classes of phenomena that mathematics has been created to analyse. These mathematical content categories (*quantity, uncertainty and data, change and relationships,* and *space and shape*) are also identified in the outer-most box of Figure 1.1.

- To solve such contextualised problems, individuals must apply mathematical thought and action to the challenge, and the framework characterises this in three different ways. First, Figure 1.1 acknowledges the need of the individual to draw upon a variety of mathematical concepts, knowledge and skills during the work. This mathematical knowledge is drawn upon as the individual represents and communicates mathematics, devises strategies, reasons and makes arguments, and so forth. These mathematical actions are characterised in the framework in terms of seven fundamental mathematical capabilities which are listed in Figure 1.1 and described in detail later in the document. As an individual works on the problem – which may require problem formulation, employing mathematical concepts or procedures, or interpretation of a mathematical solution – the fundamental mathematical capabilities are activated successively and simultaneously, drawing on mathematical content from appropriate topics, to create a solution.
- The visual depiction of the mathematical modelling cycle in the inner-most box of Figure 1.1 portrays an idealised and simplified version of the stages through which a problem solver moves when exhibiting mathematical literacy. It shows an idealised series of stages that begin with the "problem in context". The problem solver tries to identify the relevant mathematics in the problem situation and *formulates* the situation mathematically according to the concepts and relationships identified and simplifying assumptions made. The problem solver thus transforms the "problem in context" into a "mathematical problem" amenable to mathematical treatment. The downward-pointing arrow in Figure 1.1 depicts the work undertaken as the problem solver *employs* mathematical concepts, procedures, facts, and tools to obtain "mathematical results". This stage typically involves mathematical reasoning, manipulation, transformation and computation. Next, the "mathematical results" need to be *interpreted* in terms of the original problem ("results in context"). This involves the problem solver interpreting, applying, and evaluating mathematical outcomes and their reasonableness in the context of a real-world-based problem. These processes of *formulating, employing,* and *interpreting* mathematics are key components of the mathematical modelling cycle and also key components of the definition of mathematical literacy. These three processes each draw on fundamental mathematical capabilities, which in turn draw on the problem solver's detailed mathematical knowledge about individual topics.

■ Figure 1.1 ■

A model of mathematical literacy in practice



The modelling cycle is a central aspect of the PISA conception of students as active problem solvers; however, it is often not necessary to engage in every stage of the modelling cycle, especially in the context of an assessment (Niss et al., 2007). It is often the case that significant parts of the mathematical modelling cycle have been undertaken by others, and the end user carries out some of the steps of the modelling cycle, but not all of them. For example, in some cases, mathematical representations, such as graphs or equations, are given that can be directly manipulated in order to answer some question or to draw some conclusion. For this reason, many PISA items involve only parts of the modelling cycle. In reality, the

problem solver may also sometimes oscillate between the processes, returning to revisit earlier decisions and assumptions. Each of the processes may present considerable challenges, and several iterations around the whole cycle may be required.

An explicit link to a variety of contexts for problems in PISA 2012

The reference to "a variety of contexts" in the definition of mathematical literacy is purposeful and intended as a way to link to the specific contexts that are described and exemplified more fully later in this framework. The specific contexts themselves are not so important, but the four categories selected for use here (personal, occupational, societal, and scientific) do reflect a wide range of situations in which individuals may meet mathematical opportunities. The definition also acknowledges that mathematical literacy helps individuals recognise the role that mathematics plays in the world and in helping them make the kinds of well-founded judgments and decisions required of constructive, engaged, and reflective citizens.

A visible role for mathematical tools, including technology in PISA 2012

The definition of mathematical literacy explicitly includes the use of mathematical tools. These tools are physical and digital equipment, software, and calculation devices.[1] Computer-based mathematical tools are in common use in workplaces of the 21st century, and will be increasingly more prevalent as the century progresses. The nature of work-related problems and logical reasoning has expanded with these new opportunities – creating enhanced expectations for mathematical literacy.

A computer-based assessment of mathematics is an area for innovation within the PISA 2012 survey, and is being offered as an option to participating countries. Reference to mathematical tools in the definition of mathematical literacy is, therefore, particularly appropriate. The use of calculators has been permitted in all PISA mathematics surveys to date, where consistent with the policy of the participating country. While previous PISA mathematics items have been developed to be as 'calculator neutral' as possible, for some of the paper-based items presented to students in 2012, a calculator may be of positive assistance; and for the optional computer-based survey component, mathematical tools such as an online calculator will be included as part of the computer-based test material provided for some questions. Since PISA items reflect problems that arise in personal, occupational, societal, and scientific contexts, and calculators are used in all of these settings, a calculator is of assistance in some PISA items. A computer-based assessment will provide the opportunity to include a wider range of mathematics tools – such as statistical software, geometric construction and visualisation utilities, and virtual measuring instruments – into the assessment items. This will reflect the medium that increasingly more individuals use for interfacing with their world and for solving problems, and it also will provide the opportunity to assess some aspects of mathematical literacy that are not easily assessed via traditional paper-based tests.

ORGANISING THE DOMAIN

The PISA mathematics framework defines the domain of mathematics for the PISA survey and describes an approach to the assessment of the mathematical literacy of 15-year-olds. That is, PISA assesses the extent to which 15-year-old students can handle mathematics adeptly when confronted with situations and problems – the majority of which are presented in real-world contexts.

For purposes of the assessment, the PISA 2012 definition of mathematical literacy can be analysed in terms of three interrelated aspects:

- the mathematical *processes* that describe what individuals do to connect the context of the problem with mathematics and thus solve the problem, and the capabilities that underlie those processes;
- the mathematical *content* that is targeted for use in the assessment items; and
- the *contexts* in which the assessment items are located.

The following sections elaborate these aspects. In highlighting these aspects of the domain, the PISA 2012 mathematics framework helps ensure that assessment items developed for the survey reflect a range of processes, content, and contexts, so that, considered as a whole, the set of assessment items effectively operationalises what this framework defines as mathematical literacy. Several questions, based on the PISA 2012 definition of mathematical literacy lie behind the organisation of this section of the framework. They are:

- What processes do individuals engage in when solving contextual mathematical problems, and what capabilities do we expect individuals to be able to demonstrate as their mathematical literacy grows?
- What mathematical content knowledge can we expect of individuals – and of 15-year-old students in particular?
- In what contexts is mathematical literacy able to be observed and assessed?

Mathematical processes and the underlying mathematical capabilities

Mathematical processes

The definition of mathematical literacy refers to an individual's capacity to *formulate, employ*, and *interpret* mathematics. These three words, "formulate", "employ" and "interpret", provide a useful and meaningful structure for organising the mathematical processes that describe what individuals do to connect the context of a problem with the mathematics and thus solve the problem. The PISA 2012 mathematics survey will, for the first time, report results according to these mathematical processes, and this structure will provide useful and policy-relevant categories when reporting results. The categories to be used for reporting are as follows:

- *formulating* situations mathematically;
- *employing* mathematical concepts, facts, procedures, and reasoning; and
- *interpreting*, applying and evaluating mathematical outcomes.

It is important for both policy makers and those engaged more closely in the day-to-day education of students to know how effectively students are able to engage in each of these processes. The results of the PISA survey for the formulating process indicate how effectively students are able to recognise and identify opportunities to use mathematics in problem situations and then provide the necessary mathematical structure needed to formulate that contextualised problem into a mathematical form. The results of the PISA survey for the employing process indicate how well students are able to perform computations and manipulations and apply the concepts and facts that they know to arrive at a mathematical solution to a problem formulated mathematically. The results of the PISA survey for the interpreting process indicate how effectively students are able to reflect upon mathematical solutions or conclusions, interpret them in the context of a real-world problem, and determine whether the results or conclusions are reasonable. Students' facility at applying mathematics to problems and situations is dependent on skills inherent in all three of these processes, and an understanding of their effectiveness in each category can help inform both policy-level discussions and decisions being made closer to the classroom level.

Formulating situations mathematically

The word "formulate" in the mathematical literacy definition refers to individuals being able to recognise and identify opportunities to use mathematics and then provide mathematical structure to a problem presented in some contextualised form. In the process of *formulating situations mathematically*, individuals determine where they can extract the essential mathematics to analyse, set up, and solve the problem. They translate from a real-world setting to the domain of mathematics and provide the real-world problem with mathematical structure, representations, and specificity. They reason about and make sense of constraints and assumptions in the problem. Specifically, this process of *formulating situations mathematically* includes activities such as the following:

- identifying the mathematical aspects of a problem situated in a real-world context and identifying the significant variables;
- recognising mathematical structure (including regularities, relationships, and patterns) in problems or situations;
- simplifying a situation or problem in order to make it amenable to mathematical analysis;
- identifying constraints and assumptions behind any mathematical modelling and simplifications gleaned from the context;
- representing a situation mathematically, using appropriate variables, symbols, diagrams, and standard models;
- representing a problem in a different way, including organising it according to mathematical concepts and making appropriate assumptions;
- understanding and explaining the relationships between the context-specific language of a problem and the symbolic and formal language needed to represent it mathematically;
- translating a problem into mathematical language or a representation;
- recognising aspects of a problem that correspond with known problems or mathematical concepts, facts, or procedures; and
- using technology (such as a spreadsheet or the list facility on a graphing calculator) to portray a mathematical relationship inherent in a contextualised problem.

The released PISA item *PIZZAS* (see "Illustrative PISA mathematics items" at the end of this chapter) calls most heavily on students' abilities to formulate a situation mathematically. While it is indeed the case that students are also called upon to perform calculations as they solve the problem and make sense of the results of their calculations by identifying which pizza is the better value for the money, the real cognitive challenge of this item lies in being able to formulate a mathematical model that encapsulates the concept of value for money. The problem solver must recognise that because the pizzas have the same thickness but different diameters, the focus of the analysis can be on the area of the circular surface of the pizza. The relationship between amount of pizza and amount of money is then captured in the concept of value for money, modelled as cost per unit of area. The released PISA item *ROCK CONCERT* (see "Illustrative PISA mathematics items" at the end of this chapter) is another example of an item that relies most heavily on students' abilities to formulate a situation mathematically, as it calls on students to make sense of the contextual information provided (e.g. field size and shape, the fact that the rock concert is full, and the fact that fans are standing) and translate that information into a useful mathematical form in order to estimate the number of people attending the concert.

Employing mathematical concepts, facts, procedures and reasoning

The word "employ" in the mathematical literacy definition refers to individuals being able to apply mathematical concepts, facts, procedures, and reasoning to solve mathematically-formulated problems to obtain mathematical conclusions. In the process of *employing mathematical concepts, facts, procedures and reasoning* to solve problems, individuals perform the mathematical procedures needed to derive results and find a mathematical solution (e.g. performing arithmetic computations, solving equations, making logical deductions from mathematical assumptions, performing symbolic manipulations, extracting mathematical information from tables and graphs, representing and manipulating shapes in space, and analysing data). They work on a model of the problem situation, establish regularities, identify connections between mathematical entities, and create mathematical arguments. Specifically, this process of *employing mathematical concepts, facts, procedures, and reasoning* includes activities such as:

- devising and implementing strategies for finding mathematical solutions;
- using mathematical tools, including technology, to help find exact or approximate solutions;
- applying mathematical facts, rules, algorithms, and structures when finding solutions;
- manipulating numbers, graphical and statistical data and information, algebraic expressions and equations, and geometric representations;
- making mathematical diagrams, graphs, and constructions and extracting mathematical information from them;
- using and switching between different representations in the process of finding solutions;
- making generalisations based on the results of applying mathematical procedures to find solutions; and
- reflecting on mathematical arguments and explaining and justifying mathematical results.

The released PISA unit *WALKING* (see "Illustrative PISA mathematics items" at the end of this chapter) exemplifies items that rely most heavily on students' abilities for *employing mathematical concepts, facts, procedures, and reasoning*. Both items in this unit depend upon employing a given model – a formula – to determine either the pace length (Question 1) or walking speed (Question 2). Both questions have been expressed in terms that already have mathematical structure, and students are required to perform algebraic manipulations and calculations in order to derive solutions. Similarly, the released PISA item *CARPENTER* (see "Illustrative PISA mathematics items" at the end of this chapter) relies most heavily on students *employing mathematical concepts, facts, procedures, and reasoning*. The major cognitive challenge is to devise a strategy to find information about the total length of line segments of individually unknown lengths and to reason about the comparative lengths. Individuals also have to relate the diagrams to the gardens and the perimeters to the amount of timber available, but this process of formulating is considerably less demanding than the process of reasoning about the perimeter lengths.

Interpreting, applying and evaluating mathematical outcomes

The word "interpret" used in the mathematical literacy definition focuses on the abilities of individuals to reflect upon mathematical solutions, results, or conclusions and interpret them in the context of real-life problems. This involves translating mathematical solutions or reasoning back into the context of a problem and determining whether the results are reasonable and make sense in the context of the problem. This mathematical process category encompasses both the "interpret" and "evaluate" arrows noted in the previously defined model of mathematical literacy in practice (see Figure 1.1). Individuals engaged in this process may be called upon to construct and communicate explanations and arguments in the context of the problem, reflecting on both the modelling process and its results. Specifically, this process of *interpreting, applying, and evaluating mathematical outcomes* includes activities such as:

- interpreting a mathematical result back into the real world context;
- evaluating the reasonableness of a mathematical solution in the context of a real-world problem;
- understanding how the real world impacts the outcomes and calculations of a mathematical procedure or model in order to make contextual judgments about how the results should be adjusted or applied;

- explaining why a mathematical result or conclusion does, or does not, make sense given the context of a problem;
- understanding the extent and limits of mathematical concepts and mathematical solutions; and
- critiquing and identifying the limits of the model used to solve a problem.

The released PISA item *LITTER* (see "Illustrative PISA mathematics items" at the end of this chapter) calls most heavily on students' capacity for *interpreting, applying, and evaluating* mathematical outcomes. The focus of this item is on evaluating the effectiveness of the mathematical outcome – in this case an imagined or sketched bar graph – in portraying the data presented in the item on the decomposition time of several types of litter. The item involves reasoning about the data presented, thinking mathematically about the relationship between the data and their presentation, and evaluating the result. The problem solver must and provide a reason why a bar graph is unsuitable for displaying the provided data.

Fundamental mathematical capabilities underlying the mathematical processes

A decade of experience in developing PISA items and analysing the ways in which students respond to items has revealed that there is a set of fundamental mathematical capabilities that underpins each of these reported processes and mathematical literacy in practice. The work of Mogens Niss and his Danish colleagues (Niss, 2003; Niss and Jensen, 2002; Niss and Højgaard, 2011) identified eight capabilities – referred to as "competencies" by Niss and in the 2003 framework (OECD, 2003) – that are instrumental to mathematical behaviour. The PISA 2012 framework uses a modified formulation of this set of capabilities, which condenses the number from eight to seven based on the MEG's investigation of the operation of the competencies through previously administered PISA items (Turner et al., forthcoming). There is wide recognition of the need to identify such a set of general mathematical capabilities, to complement the role of specific mathematical content knowledge in mathematics learning. Prominent examples include the eight mathematical practices of the Common Core State Standards in the United States (2010), the four key processes (representing, analysing, interpreting and evaluating, and communicating and reflecting) of the England's Mathematics National Curriculum (Qualifications and Curriculum Authority, 2007), and the process standards in the National Council of Teachers of Mathematics (NCTM) "Principles and Standards for School Mathematics" (NCTM, 2000). These cognitive capabilities are available to or learnable by individuals in order to understand and engage with the world in a mathematical way, or to solve problems. As the level of mathematical literacy possessed by an individual increases, that individual is able to draw to an increasing degree on the fundamental mathematical capabilities (Turner and Adams, 2012). Thus, increasing activation of fundamental mathematical capabilities is associated with increasing item difficulty. This observation has been used as the basis of the descriptions of different proficiency levels of mathematical literacy reported in previous PISA surveys and discussed later in this framework, in Box 1.1.

The seven fundamental mathematical capabilities used in this framework are as follows:

- *Communication*: Mathematical literacy involves *communication*. The individual perceives the existence of some challenge and is stimulated to recognise and understand a problem situation. Reading, decoding and interpreting statements, questions, tasks or objects enables the individual to form a mental model of the situation, which is an important step in understanding, clarifying and formulating a problem. During the solution process, intermediate results may need to be summarised and presented. Later on, once a solution has been found, the problem solver may need to present the solution, and perhaps an explanation or justification, to others.
- *Mathematising*: Mathematical literacy can involve transforming a problem defined in the real world to a strictly mathematical form (which can include structuring, conceptualising, making assumptions, and/or formulating a model), or interpreting or evaluating a mathematical outcome or a mathematical model in relation to the original problem. The term "mathematising" is used to describe the fundamental mathematical activities involved.
- *Representation*: Mathematical literacy very frequently involves *representations* of mathematical objects and situations. This can entail selecting, interpreting, translating between, and using a variety of representations to capture a situation, interact with a problem, or to present one's work. The representations referred to include graphs, tables, diagrams, pictures, equations, formulae, and concrete materials.
- *Reasoning and argument*: A mathematical ability that is called on throughout the different stages and activities associated with mathematical literacy is referred to as *reasoning and argument*. This capability involves logically rooted thought processes that explore and link problem elements so as to make inferences from them, check a justification that is given, or provide a justification of statements or solutions to problems.
- *Devising strategies for solving problems*: Mathematical literacy frequently requires *devising strategies for solving problems* mathematically. This involves a set of critical control processes that guide an individual to effectively recognise, formulate and solve problems. This skill is characterised as selecting or devising a plan or strategy to use mathematics

to solve problems arising from a task or context, as well as guiding its implementation. This mathematical capability can be demanded at any of the stages of the problem-solving process.

- *Using symbolic, formal and technical language and operations*: Mathematical literacy requires *using symbolic, formal and technical language and operations*. This involves understanding, interpreting, manipulating, and making use of symbolic expressions within a mathematical context (including arithmetic expressions and operations) governed by mathematical conventions and rules. It also involves understanding and utilising formal constructs based on definitions, rules and formal systems and also using algorithms with these entities. The symbols, rules and systems used will vary according to what particular mathematical content knowledge is needed for a specific task to formulate, solve or interpret the mathematics.
- *Using mathematical tools*: The final mathematical capability that underpins mathematical literacy in practice is *using mathematical tools*. Mathematical tools encompass physical tools such as measuring instruments, as well as calculators and computer-based tools that are becoming more widely available. This ability involves knowing about and being able to make use of various tools that may assist mathematical activity, and knowing about the limitations of such tools. Mathematical tools can also have an important role in communicating results. Previously it has been possible to include the use of tools in paper-based PISA surveys in only a very minor way. The optional computer-based component of the PISA 2012 mathematics assessment will provide more opportunities for students to use mathematical tools and to include observations about the way tools are used as part of the assessment.

These capabilities are evident to varying degrees in each of the three mathematical processes to be used for reporting purposes. The ways in which these capabilities manifest themselves within the three processes are described in Figure 1.2. More detail on these capabilities, particularly as they relate to item difficulty, can be found at the end of this chapter in the Box 1.1 “Fundamental mathematical capabilities and their relationship to item difficulty”. In addition, each of the illustrative examples provided in the section “Illustrative PISA mathematics items” describes how the capabilities might be activated by students solving that particular problem.

Mathematical content knowledge

An understanding of mathematical content – and the ability to apply that knowledge to the solution of meaningful contextualised problems – is important for citizens in the modern world. That is, to solve problems and interpret situations in personal, occupational, societal and scientific contexts, there is a need to draw upon certain mathematical knowledge and understandings.

Mathematical structures have been developed over time as a means to understand and interpret natural and social phenomena. In schools, the mathematics curriculum is typically organised around content strands (e.g. number, algebra and geometry) and detailed topic lists that reflect historically well-established branches of mathematics and that help in defining a structured curriculum. However, outside the mathematics classroom, a challenge or situation that arises is usually not accompanied by a set of rules and prescriptions that shows how the challenge can be met. Rather it typically requires some creative thought in seeing the possibilities of bringing mathematics to bear on the situation and in formulating it mathematically. Often a situation can be addressed in different ways drawing on different mathematical concepts, procedures, facts or tools.

Since the goal of PISA is to assess mathematical literacy, an organisational structure for mathematical content knowledge is proposed based on the mathematical phenomena that underlie broad classes of problems and which have motivated the development of specific mathematical concepts and procedures. For example, mathematical phenomena such as uncertainty and change underlie many commonly occurring situations, and mathematical strategies and tools have been developed to analyse such situations. Such an organisation for content is not new, as exemplified by two well-known publications: *On the Shoulders of Giants: New Approaches to Numeracy* (Steen, 1990) and *Mathematics: The Science of Patterns* (Devlin, 1994).

Because national mathematics curricula are typically designed to equip students with knowledge and skills that address these same underlying mathematical phenomena, the outcome is that the range of content arising from organising content this way is closely aligned with that typically found in national mathematics curricula. For guidance to item writers, this framework also lists some content topics appropriate for assessing the mathematical literacy of 15-year-old students, based on analyses of national standards from eleven countries.[2]

To organise the domain of mathematics for purposes of assessing mathematical literacy, it is important to select a structure that grows out of historical developments in mathematics, that encompasses sufficient variety and depth to reveal the essentials of mathematics, and that also represents, or includes, the conventional mathematical strands in an

■ Figure 1.2 ■

Relationship between mathematical processes and fundamental mathematical capabilities

	Formulating **situations mathematically**	***Employing*** **mathematical concepts, facts, procedures and reasoning**	***Interpreting*, applying and evaluating mathematical outcomes**
Communicating	Read, decode, and make sense of statements, questions, tasks, objects, images, or animations (in computer-based assessment) in order to form a mental model of the situation	Articulate a solution, show the work involved in reaching a solution and/or summarise and present intermediate mathematical results	Construct and communicate explanations and arguments in the context of the problem
Mathematising	Identify the underlying mathematical variables and structures in the real world problem, and make assumptions so that they can be used	Use an understanding of the context to guide or expedite the mathematical solving process, e.g. working to a context-appropriate level of accuracy	Understand the extent and limits of a mathematical solution that are a consequence of the mathematical model employed
Representation	Create a mathematical representation of real-world information	Make sense of, relate and use a variety of representations when interacting with a problem	Interpret mathematical outcomes in a variety of formats in relation to a situation or use; compare or evaluate two or more representations in relation to a situation
Reasoning and argument	Explain, defend or provide a justification for the identified or devised representation of a real-world situation	Explain, defend or provide a justification for the processes and procedures used to determine a mathematical result or solution Connect pieces of information to arrive at a mathematical solution, make generalisations or create a multi-step argument	Reflect on mathematical solutions and create explanations and arguments that support, refute or qualify a mathematical solution to a contextualised problem
Devising strategies for solving problems	Select or devise a plan or strategy to mathematically reframe contextualised problems	Activate effective and sustained control mechanisms across a multi-step procedure leading to a mathematical solution, conclusion, or generalisation	Devise and implement a strategy in order to interpret, evaluate and validate a mathematical solution to a contextualised problem
Using symbolic, formal and technical language and operations	Use appropriate variables, symbols, diagrams and standard models in order to represent a real-world problem using symbolic/formal language	Understand and utilise formal constructs based on definitions, rules and formal systems as well as employing algorithms	Understand the relationship between the context of the problem and representation of the mathematical solution. Use this understanding to help interpret the solution in context and gauge the feasibility and possible limitations of the solution
Using mathematical tools	Use mathematical tools in order to recognise mathematical structures or to portray mathematical relationships	Know about and be able to make appropriate use of various tools that may assist in implementing processes and procedures for determining mathematical solutions	Use mathematical tools to ascertain the reasonableness of a mathematical solution and any limits and constraints on that solution, given the context of the problem

acceptable way. Historically, with the 17th century invention of analytic geometry and calculus, mathematics became an integrated study of number, shape, change, and relationships; analysis of such phenomena as randomness and indeterminacy became instrumental to problem solving in the 19th and 20th centuries. Thus, a set of *content categories* that reflects the range of underlying mathematical phenomena was selected for the PISA 2012 framework, consistent with the categories used for previous PISA surveys.

The following list of content categories, therefore, is used in PISA 2012 to meet the requirements of historical development, coverage of the domain of mathematics and the underlying phenomena which motivate its development, and reflection of the major strands of school curricula. These four categories characterise the range of mathematical content that is central to the discipline and illustrate the broad areas of content that guide development of test items for PISA 2012:

- *change and relationships;*
- *space and shape;*
- *quantity;* and
- *uncertainty and data.*[3]

With these four categories, the mathematical domain can be organised in a way that ensures a spread of items across the domain and focuses on important mathematical phenomena, but at the same time, avoids a too fine division that would work against a focus on rich and challenging mathematical problems based on real situations. While categorisation by content category is important for item development and selection, and for reporting of assessment results, it is important to note that some specific content topics may materialise in more than one content category. For example, a released PISA item called *PIZZAS* involves determining which of two round pizzas, with different diameters and different costs but the same thickness, is the better value (see "Illustrative PISA mathematics items" at the end of this chapter to view this item and an analysis of its attributes). This item draws on several areas of mathematics, including measurement, quantification (value for money, proportional reasoning and arithmetic calculations), and change and relationships (in terms of relationships among the variables and how relevant properties change from the smaller pizza to the larger one). This item was ultimately categorised as a *change and relationships* item since the key to the problem lies in students being able to relate the change in areas of the two pizzas (given a change in diameter) and a corresponding change of price. Clearly, a different item involving circle area might be classified as a *space and shape* item. Connections between aspects of content that span these four content categories contribute to the coherence of mathematics as a discipline and are apparent in some of the assessment items selected for the PISA 2012 assessment.

The broad mathematical content categories and the more specific content topics appropriate for 15-year-old students described later in this section reflect the level and breadth of content that is eligible for inclusion on the PISA 2012 survey. Narrative descriptions of each content category and the relevance of each to solving meaningful problems are provided first, followed by more specific definitions of the kinds of content that are appropriate for inclusion in an assessment of mathematical literacy of 15-year-old students. These specific topics reflect commonalities found in the expectations set by a range of countries and educational jurisdictions. The standards examined to identify these content topics are viewed as evidence not only of what is taught in mathematics classrooms in these countries but also as indicators of what countries view as important knowledge and skills for preparing students of this age to become constructive, engaged and reflective citizens.

Descriptions of the mathematical content knowledge that characterise each of the four categories – *change and relationships, space and shape, quantity* and *uncertainty and data* – are provided below.

Change and relationships

The natural and designed worlds display a multitude of temporary and permanent relationships among objects and circumstances, where changes occur within systems of interrelated objects or in circumstances where the elements influence one another. In many cases these changes occur over time, and in other cases changes in one object or quantity are related to changes in another. Some of these situations involve discrete change; others change continuously. Some relationships are of a permanent, or invariant, nature. Being more literate about change and relationships involves understanding fundamental types of change and recognising when they occur in order to use suitable mathematical models to describe and predict change. Mathematically this means modelling the change and the relationships with appropriate functions and equations, as well as creating, interpreting, and translating among symbolic and graphical representations of relationships.

Change and relationships is evident in such diverse settings as growth of organisms, music, the cycle of seasons, weather patterns, employment levels and economic conditions. Aspects of the traditional mathematical content of functions and algebra, including algebraic expressions, equations and inequalities, tabular and graphical representations, are central in describing, modelling, and interpreting change phenomena. For example, the released PISA unit *WALKING* (see "Illustrative PISA mathematics items" at the end of this chapter) contains two items that exemplify the *change and relationships* category since the focus is on the algebraic relationships between two variables, requiring students to activate their algebraic knowledge and skills. Students are required to employ a given formula for pace length – a formula expressed in algebraic form – to determine pace length in one item and walking speed in the other. Representations of data and relationships described using statistics also are often used to portray and interpret change and relationships, and a firm grounding in the basics of number and units is also essential to defining and interpreting *change and relationships*. Some interesting relationships arise from geometric measurement, such as the way that changes in perimeter of a family of shapes might relate to changes in area, or the relationships among lengths of the sides of triangles. The released PISA item *PIZZAS* (see "Illustrative PISA mathematics items" at the end of this chapter) exemplifies *change and relationships*.

The optional computer-based assessment of mathematics in 2012 makes it possible to present students with dynamic images, multiple representations that are dynamically linked, and the opportunity to manipulate functions. For example, change over time (e.g. growth or movement) can be directly depicted in animations and simulations, and represented by linked functions, graphs and tables of data. Finding and using mathematical models of change is enhanced when individuals can explore and describe change by working with software that can graph functions, manipulate parameters, produce tables of values, experiment with geometric relationships, organise and plot data, and calculate with formulas. The capability of spreadsheets and graphing utilities to work with formulas and plot data is especially relevant.

Space and shape

Space and shape encompasses a wide range of phenomena that are encountered everywhere in our visual and physical world: patterns, properties of objects, positions and orientations, representations of objects, decoding and encoding of visual information, navigation and dynamic interaction with real shapes as well as with representations. Geometry serves as an essential foundation for *space and shape*, but the category extends beyond traditional geometry in content, meaning and method, drawing on elements of other mathematical areas such as spatial visualisation, measurement and algebra. For instance, shapes can change, and a point can move along a locus, thus requiring function concepts. Measurement formulas are central in this area. The manipulation and interpretation of shapes in settings that call for tools ranging from dynamic geometry software to Global Positioning System (GPS) software are included in this content category.

PISA assumes that the understanding of a set of core concepts and skills is important to mathematical literacy relative to *space and shape*. Mathematical literacy in the area of *space and shape* involves a range of activities such as understanding perspective (for example in paintings), creating and reading maps, transforming shapes with and without technology, interpreting views of three-dimensional scenes from various perspectives and constructing representations of shapes. The released PISA item *CARPENTER* (see "Illustrative PISA mathematics items" at the end of this chapter) belongs to this category since it deals with another key aspect of *space and shape* – properties of shapes. In this complex multiple-choice item, students are presented with four different designs for a garden bed and asked which one(s) can be edged with 32 metres of timber. This item requires the application of geometrical knowledge and reasoning. Enough information is given to enable direct calculation of the exact perimeter for three of the designs; however, inexact information is given for one design, meaning that students need to employ qualitative geometric reasoning skills.

Computer-based assessment provides students with the opportunity to manipulate dynamic representations of shapes and explore relationships within and among geometrical objects in three dimensions, which can be virtually rotated to promote an accurate mental image. Students can work with maps where zooming and rotation are possible to build up a mental picture of a place and use such tools to assist in planning routes. They can choose and use virtual tools to make measurements (e.g., of angles and line segments) on plans, images and models, and use the data in calculations. Technology allows students to integrate knowledge of geometry with visual information to build an accurate mental model. For example, to find the volume of a cup, an individual might manipulate the image to identify that it is a truncated cone, to identify the perpendicular height and where it may be measured, and to ascertain that what might look like ellipses at the top and bottom in a two-dimensional picture are actually circles in three-dimensional space.

Quantity

The notion of *quantity* may be the most pervasive and essential mathematical aspect of engaging with, and functioning in, our world. It incorporates the quantification of attributes of objects, relationships, situations and entities in the world,

understanding various representations of those quantifications, and judging interpretations and arguments based on quantity. To engage with the quantification of the world involves understanding measurements, counts, magnitudes, units, indicators, relative size, and numerical trends and patterns. Aspects of quantitative reasoning – such as number sense, multiple representations of numbers, elegance in computation, mental calculation, estimation and assessment of reasonableness of results – are the essence of mathematical literacy relative to *quantity*.

Quantification is a primary method for describing and measuring a vast set of attributes of aspects of the world. It allows for the modelling of situations, for the examination of change and relationships, for the description and manipulation of space and shape, for organising and interpreting data, and for the measurement and assessment of uncertainty. Thus mathematical literacy in the area of *quantity* applies knowledge of number and number operations in a wide variety of settings. The released PISA item *ROCK CONCERT* (see "Illustrative PISA mathematics items" at the end of this chapter) is an item exemplifying the *quantity* category. This item asks students to estimate the total number of people attending a concert, given the dimensions of the rectangular field reserved for the concert. While this item also has some elements that relate to the *space and shape* category, its primary demand comes from postulating a reasonable area for each person and using the total area available to calculate an estimated number of people attending. Alternately, given that this item is multiple-choice, students might work backwards using the area of the field and each of the response options to calculate the corresponding space per person, determining which provides the most reasonable result. Since response options are provided in terms of thousands (e.g. 2000, 5000) this item also calls on students' numerical estimation skills.

Computer-based assessment provides students with the opportunity to take advantage of the vast computational power of modern technology. It is important to note that while technology can relieve the burden of computation from individuals and free some cognitive resources to focus on meaning and strategy when solving problems, this does not remove the need for mathematically literate individuals to have a deep understanding of mathematics. An individual without such an understanding can at best use technology for routine tasks only, which is not consistent with the PISA 2012 definition of mathematical literacy. Moreover, integration of technology into the optional computer-based assessment allows for the inclusion of items that call for levels of numeric and statistical calculation that are unmanageable in the paper-based assessment.

Uncertainty and data

In science, technology and everyday life, uncertainty is a given. Uncertainty is therefore a phenomenon at the heart of the mathematical analysis of many problem situations, and the theory of probability and statistics as well as techniques of data representation and description have been established to deal with it. The *uncertainty and data* content category includes recognising the place of variation in processes, having a sense of the quantification of that variation, acknowledging uncertainty and error in measurement, and knowing about chance. It also includes forming, interpreting and evaluating conclusions drawn in situations where uncertainty is central. The presentation and interpretation of data are key concepts in this category (Moore, 1997).

There is uncertainty in scientific predictions, poll results, weather forecasts, and economic models. There is variation in manufacturing processes, test scores and survey findings, and chance is fundamental to many recreational activities enjoyed by individuals. The traditional curricular areas of probability and statistics provide formal means of describing, modelling and interpreting a certain class of uncertainty phenomena, and for making inferences. In addition, knowledge of number and of aspects of algebra such as graphs and symbolic representation contribute to facility in engaging in problems in this content category. The released PISA item *LITTER* (see "Illustrative PISA mathematics items" at the end of this chapter) is categorised as dealing with *uncertainty and data*. This item requires students to examine data presented in a table and explain why a bar graph is not suitable for displaying these data. The focus on the interpretation and presentation of data is an important aspect of the *uncertainty and data* category.

Computer-based assessment gives students the opportunity to work with larger data sets and provides the computational power and data handling capacities they need to work with such sets. Students are given the opportunity to choose appropriate tools to manipulate, analyse and represent data, and to sample from data populations. Linked representations allow students to examine and describe such data in different ways. The capacity to generate random outcomes, including numbers, enables probabilistic situations to be explored using simulations, such as the empirical likelihood of events and properties of samples.

Content topics for guiding the assessment of mathematical literacy for 15-year-old students

To effectively understand and solve contextualised problems involving *change and relationships, space and shape, quantity* and *uncertainty and data* requires drawing upon a variety of mathematical concepts, procedures, facts, and tools at an appropriate level of depth and sophistication. As an assessment of mathematical literacy, PISA strives to assess

the levels and types of mathematics that are appropriate for 15-year-old students on a trajectory to become constructive, engaged and reflective citizens able to make well-founded judgments and decisions. It is also the case that PISA, while not designed or intended to be a curriculum-driven assessment, strives to reflect the mathematics that students have likely had the opportunity to learn by the time they are 15 years old.

With an eye toward developing an assessment that is both forward-thinking yet reflective of the mathematics that 15-year-old students have likely had the opportunity to learn, analyses were conducted of a sample of mathematics standards from eleven countries to determine both what is being taught to students in classrooms around the world and also what countries deem realistic and important preparation for students as they approach entry into the workplace or admission into a higher education institution. Based on commonalities identified in these analyses, coupled with the judgment of mathematics experts, content deemed appropriate for inclusion in the assessment of mathematical literacy of 15-year-old students on PISA 2012 is described below.

The four content categories of *change and relationships, space and shape, quantity* and *uncertainty and data* serve as the foundation for identifying this range of content, yet there is not a one-to-one mapping of content topics to these categories. For example, proportional reasoning comes into play in such varied contexts as making measurement conversions, analysing linear relationships, calculating probabilities and examining the lengths of sides in similar shapes. The following content is intended to reflect the centrality of many of these concepts to all four content categories and reinforce the coherence of mathematics as a discipline. It intends to be illustrative of the content topics included in PISA 2012, rather than an exhaustive listing:

- *Functions*: The concept of function, emphasising but not limited to linear functions, their properties, and a variety of descriptions and representations of them. Commonly used representations are verbal, symbolic, tabular and graphical.
- *Algebraic expressions*: Verbal interpretation of and manipulation with algebraic expressions, involving numbers, symbols, arithmetic operations, powers and simple roots.
- *Equations and inequalities*: Linear and related equations and inequalities, simple second-degree equations, and analytic and non-analytic solution methods.
- *Co-ordinate systems*: Representation and description of data, position and relationships.
- *Relationships within and among geometrical objects in two and three dimensions*: Static relationships such as algebraic connections among elements of figures (e.g. the Pythagorean Theorem as defining the relationship between the lengths of the sides of a right triangle), relative position, similarity and congruence, and dynamic relationships involving transformation and motion of objects, as well as correspondences between two- and three-dimensional objects.
- *Measurement*: Quantification of features of and among shapes and objects, such as angle measures, distance, length, perimeter, circumference, area and volume.
- *Numbers and units*: Concepts, representations of numbers and number systems, including properties of integer and rational numbers, relevant aspects of irrational numbers, as well as quantities and units referring to phenomena such as time, money, weight, temperature, distance, area and volume, and derived quantities and their numerical description.
- *Arithmetic operations*: The nature and properties of these operations and related notational conventions.
- *Percents, ratios and proportions*: Numerical description of relative magnitude and the application of proportions and proportional reasoning to solve problems.
- *Counting principles*: Simple combinations and permutations.
- *Estimation*: Purpose-driven approximation of quantities and numerical expressions, including significant digits and rounding.
- *Data collection, representation and interpretation*: Nature, genesis and collection of various types of data, and the different ways to represent and interpret them.
- *Data variability and its description*: Concepts such as variability, distribution and central tendency of data sets, and ways to describe and interpret these in quantitative terms.
- *Samples and sampling*: Concepts of sampling and sampling from data populations, including simple inferences based on properties of samples.
- *Chance and probability*: Notion of random events, random variation and its representation, chance and frequency of events, and basic aspects of the concept of probability.

Contexts

An important aspect of mathematical literacy is that mathematics is engaged in solving a problem set in a context. The context is the aspect of an individual's world in which the problems are placed. The choice of appropriate mathematical strategies and representations is often dependent on the context in which a problem arises. Being able to work within a context is widely appreciated to place additional demands on the problem solver (see Watson and Callingham, 2003, for findings about statistics). For the PISA survey, it is important that a wide variety of contexts are used. This offers the possibility of connecting with the broadest possible range of individual interests and with the range of situations in which individuals operate in the 21st century.

For purposes of the PISA 2012 mathematics framework, four context categories have been defined and are used to classify assessment items developed for the PISA survey:

- *Personal*: Problems classified in the personal context category focus on activities of one's self, one's family or one's peer group. The kinds of contexts that may be considered personal include (but are not limited to) those involving food preparation, shopping, games, personal health, personal transportation, sports, travel, personal scheduling and personal finance. The released PISA item *PIZZAS* (see "Illustrative PISA mathematics items" at the end of this chapter) is set in a personal context since the question posed by the item is which pizza provides the purchaser with the better value for the money. Similarly, the released PISA unit *WALKING* (see "Illustrative PISA mathematics items" at the end of this chapter) contains two items that reflect a personal context. The first item involves applying a mathematical formula to determine of an individual's pace length, while the second item involves the application of the same formula to determine of another individual's walking speed.
- *Occupational*: Problems classified in the occupational context category are centred on the world of work. Items categorised as occupational may involve (but are not limited to) such things as measuring, costing and ordering materials for building, payroll/accounting, quality control, scheduling/inventory, design/architecture and job-related decision making. Occupational contexts may relate to any level of the workforce, from unskilled work to the highest levels of professional work, although items in the PISA survey must be accessible to 15-year-old students. The released PISA item *CARPENTER* (see "Illustrative PISA mathematics items" at the end of this chapter) is categorised as occupational as it deals with a work task of a carpenter to construct a border around a garden bed. An item requiring similar mathematical analysis to the *PIZZAS* item discussed earlier, which presented the situation from the point of view of the pizza seller instead of the purchaser, would be placed in the occupational category.
- *Societal*: Problems classified in the societal context category focus on one's community (whether local, national or global). They may involve (but are not limited to) such things as voting systems, public transport, government, public policies, demographics, advertising, national statistics and economics. Although individuals are involved in all of these things in a personal way, in the societal context category the focus of problems is on the community perspective. The released PISA item *ROCK CONCERT* (see "Illustrative PISA mathematics items" at the end of this chapter) is an example of an item categorised as societal since it is set at the level of the rock concert organisation, even though it draws on the personal experience of being in crowds.
- *Scientific*: Problems classified in the scientific category relate to the application of mathematics to the natural world and issues and topics related to science and technology. Particular contexts might include (but are not limited to) such areas as weather or climate, ecology, medicine, space science, genetics, measurement and the world of mathematics itself. The released PISA item *LITTER* (see "Illustrative PISA mathematics items" at the end of this chapter) is an example of an item set in a scientific context, since its focus is related to scientific issues pertaining to the environment, and specifically to data on decomposition time. Items that are intramathematical, where all the elements involved belong in the world of mathematics, fall within the scientific context.

PISA assessment items are arranged in units that share stimulus material. It is therefore usually the case that all items in the same unit belong to the same context category. Exceptions do arise; for example stimulus material may be examined from a personal point of view in one item and a societal point of view in another. When an item involves only mathematical constructs without reference to the contextual elements of the unit within which it is located, it is allocated to the context category of the unit. In the unusual case of a unit involving only mathematical constructs and being without reference to any context outside of mathematics, the unit is assigned to the scientific context category.

Using these context categories provides the basis for selecting a mix of item contexts and ensures that the assessment reflects a broad range of uses of mathematics, ranging from everyday personal uses to the scientific demands of global problems. Moreover it is important that each context category be populated with assessment items having a broad range of item difficulties. Given that the major purpose of these context categories is to challenge students in a broad range of

problem contexts, each category should contribute substantially to the measurement of mathematical literacy. It should not be the case that the difficulty level of assessment items representing one context category is systematically higher or lower than the difficulty level of assessment items in another category.

In identifying contexts that may be relevant, it is critical to keep in mind that a purpose of the assessment is to gauge the use of mathematical content knowledge, processes, and capabilities that students have acquired by age 15. Contexts for assessment items, therefore, are selected in light of relevance to students' interests and lives, and the demands that will be placed upon them as they enter society as constructive, engaged and reflective citizens. National project managers from countries participating in the PISA survey are involved in judging the degree of such relevance.

ASSESSING MATHEMATICAL LITERACY

In this section, the approach taken to implement the elements of the framework described in previous sections into the PISA survey for 2012, is outlined. This includes the structure of the mathematics component of the PISA survey, the reporting of levels of mathematical proficiency, the attitudes to be investigated that relate to mathematical proficiency, and arrangements for the optional computer-based survey component for mathematics.

Structure of the PISA 2012 mathematics assessment

In accordance with the definition of mathematical literacy, assessment items used in any instruments that are developed as part of the PISA survey, both paper-based and computer-based, are set within a context. Items involve the application of important mathematical concepts, knowledge, understandings and skills (mathematical content knowledge) at the appropriate level for 15-year-old students, as described earlier. The framework is used to guide the structure and content of the assessment, and it is important that the survey instruments, both paper-based and computer-based, include an appropriate balance of items reflecting the components of the mathematical literacy framework.

Desired distribution of score points by mathematical process

In addition, assessment items in the PISA 2012 mathematics survey can be assigned to one of three mathematical processes. The goal in constructing the assessment is to achieve a balance that provides approximately equal weighting between the two processes that involve making a connection between the real world and the mathematical world and the process that calls for students to be able to work on a mathematically formulated problem.

Table 1.1
Approximate distribution of score points in mathematics, by process category

Process category	Percentage of score points
Formulating situations mathematically	Approximately 25
Employing mathematical concepts, facts, procedures and reasoning	Approximately 50
Interpreting, applying and evaluating mathematical outcomes	Approximately 25
TOTAL	100

It is important to note that items in each process category should have a range of difficulty and mathematical demand.

Desired distribution of score points by content category

PISA mathematics items are selected to reflect the mathematical content knowledge described earlier in this framework. The items selected for PISA 2012 are distributed across the four content categories, as shown in Table 1.2. The goal in constructing the survey is a distribution of items with respect to content category that provides as balanced a distribution of score points as possible, since all of these domains are important for constructive, engaged and reflective citizens.

Table 1.2
Approximate distribution of score points in mathematics, by content category

Content category	Percentage of score points
Change and relationships	Approximately 25
Space and shape	Approximately 25
Quantity	Approximately 25
Uncertainty and data	Approximately 25
TOTAL	100

It is important to note that items in each content category should have a range of difficulty and mathematical demand.

Desired distribution of score points by context category

For PISA 2012, each item is set in one of four context categories. The items selected for the PISA 2012 mathematics survey represent a spread across these context categories, as described in Table 1.3. With this balanced distribution, no single context type is allowed to dominate, providing students with items that span a broad range of individual interests and a range of situations that they might expect to encounter in their lives.

Table 1.3
Approximate distribution of score points in mathematics, by context category

Context category	Percentage of score points
Personal	Approximately 25
Occupational	Approximately 25
Societal	Approximately 25
Scientific	Approximately 25
TOTAL	100

It is important to note that items in each context category should have a range of difficulty and mathematical demand.

A range of item difficulties

The PISA 2012 mathematics survey includes items with a wide range of difficulties, paralleling the range of abilities of 15-year-old students. It includes items that are challenging for the most able students, and items that are suitable for the least able students assessed in mathematics. From a psychometric perspective, a survey that is designed to measure a particular cohort of individuals is most effective and efficient when the difficulty of assessment items matches the ability of the measured subjects. Furthermore, the described proficiency scales that are used as a central part of the reporting of PISA outcomes can only include useful details for all students if the items from which the proficiency descriptions are drawn span the range of abilities described. The proficiency scales are based on increasing levels of activation of the fundamental mathematical capabilities, described fully in the Box 1.1 "Fundamental mathematical capabilities and their relationship to item difficulty". Previous PISA cycles have shown that collectively these capabilities are indicators of cognitive demand, and thus contribute centrally to item difficulty (Turner, 2012; Turner et al., forthcoming). A scale for PISA 2012 was developed after the field test and based on a description of the required activation of these capabilities. This scale provides an empirical measure of the cognitive demand for each item.

Structure of the survey instrument

The paper-based instruments for the PISA 2012 survey contain a total of 270 minutes of mathematics material. The material is arranged in nine clusters of items, with each cluster representing 30 minutes of testing time. Of this total, three clusters (representing 90 minutes of test time) comprise link material used in previous PISA surveys, four "standard" clusters (representing 120 minutes of test time) comprise new material having a wide range of difficulty, and two "easy" clusters (representing 60 minutes of test time) are devoted to material with a lower level of difficulty.

Each participating country uses seven of the clusters: the three clusters of link material, two of the new 'standard' clusters, and either the other two "standard" clusters or the two "easy" clusters. The provision of "easy" and "standard" clusters allows for better targeting of the assessment for each of the participating countries; however, the items are scaled in such a way that a country's score will not be affected if it chooses to administer either the "easy" or additional "standard" clusters. The item clusters are placed in test booklets according to a rotated test design, with each form containing four clusters of material from the mathematics, reading and science domains. Each student does one form, representing a total testing time of 120 minutes.

The optional computer-based component (CBAM) contains a total of 80 minutes of mathematics material. The material is arranged in four clusters of items, with each cluster representing 20 minutes of testing time. This material is arranged in a number of rotated test forms along with other material for computer delivery, with each form containing two clusters. Each student does one form, representing a total testing time of 40 minutes.

Design of the PISA 2012 mathematics items

Three item format types are used in the paper-based component to assess mathematical literacy in PISA 2012: open constructed-response, closed constructed-response and selected-response (multiple-choice) items. Open constructed-response items require a somewhat extended written response from a student. Such items also may ask the student to show the steps taken or to explain how the answer was reached. These items require trained experts to manually code student responses. Closed constructed-response items provide a more structured setting for presenting problem solutions, and they produce a student response that can be easily judged to be either correct or incorrect. Often student responses to questions of this type can be keyed into data capture software, and coded automatically, but some must be manually coded by trained experts. The most frequent closed constructed-responses are single numbers. Selected-response items require the choice of one or more responses from a number of response options. Responses to these questions can usually be automatically processed. About equal numbers of each of these item format types are being used to construct the survey instruments.

In the optional computer-based component, additional item format types are possible. A computer-based environment lends itself to a wider range of response modes than does paper, as well as facilitating assessment of some aspects of mathematical literacy, such as the manipulation and rotation of representations of three-dimensional shapes, which cannot so readily be assessed on paper. A computer-based assessment enables item presentation to be enhanced. For example there may be a moving stimulus, representations of three-dimensional objects that can be rotated, or more flexible access to relevant information or data. Item formats that permit a wider range of response types are also possible. For example, drag-and-drop items or the use of hot spots on an image may allow students to respond to more items non-verbally, giving a more rounded picture of mathematical literacy that is less bound to language. Some interactivity may be possible. Additionally, the possibility of automated response coding may replace some manual work; more importantly, it may facilitate coding of features in student-constructed drawings, displays and procedures that are currently impractical to code (Stacey and Wiliam, forthcoming).

The PISA mathematics survey is composed of assessment *units* comprising verbal stimulus material and often other information such as tables, charts, graphs or diagrams, plus one or more items that are linked to this common stimulus material. This format gives students the opportunity to become involved with a context or problem by responding to a series of related items. However, the measurement model used to analyse PISA data assumes item independence, so whenever units comprising more than one item are used, the objective of item writers is to ensure maximum possible independence among the items. PISA employs this unit structure to facilitate the employment of contexts that are as realistic as possible, and that reflect the complexity of real situations, while making efficient use of testing time. However, it is important to ensure that there is an adequate range of contexts so that bias due to choice of contexts is minimised and item independence is maximised. A balance between these two competing demands is therefore sought in developing the PISA survey instruments.

Items selected for inclusion in the PISA survey represent a broad range of difficulties, to match the wide ability range of students participating in the assessment. In addition, all the major categories of the assessment (the content categories, the process categories, and the context categories) are represented, to the degree possible, with items of a wide range of difficulties. Item difficulties are established as one of a number of measurement properties in an extensive field trial prior to item selection for the main PISA survey. Items are selected for inclusion in the PISA survey instruments based on their fit with framework categories and their measurement properties.

In addition, the level of reading required to successfully engage with an item is considered very carefully in item development and selection. A goal in item development is to make the wording of items as simple and direct as possible. Care is also taken to avoid item contexts that would create a cultural bias, and all choices are checked with national teams. Translation of the items into many languages is conducted very carefully, with extensive back-translation and other protocols. Attention to item bias is even more critical in PISA 2012 since the inclusion of the optional computer-based component may present new challenges to students who have not had access to computers in their mathematics classrooms.

Mathematical tools

PISA policy allows students to use calculators in the paper-based components as they are normally used in school. This represents the most authentic assessment of what students can achieve, and provides the most informative comparison of the performance of education systems. A system's choice to allow students to access and use calculators is no different, in principle, from other instructional policy decisions that are made by systems that are not controlled by PISA. In 2012, for the first time in a PISA mathematics assessment, some of the items written for paper-based delivery will be constructed in such a way that a calculator will likely make the calculations required quicker and easier – meaning that for some assessment items, it is likely that availability of a calculator will be an advantage for many students. In the paper-based component of PISA 2012, functionalities beyond the arithmetic functionality of a basic calculator will not be required.

In the optional computer-based component of PISA 2012, students will be given access to an online calculator and/ or software with equivalent functionality for items where this could be relevant. Students can also have access to a

hand-held calculator, as approved for use by 15-year-old students in their respective school systems. Other tools may also be provided as part of the test delivery system, such as virtual measuring devices, some basic spreadsheet functionality, and various graphic presentation and visualisation tools.

Item scoring

Although the majority of the items are dichotomously scored (that is, responses are awarded either credit or no credit), the open constructed-response items can sometimes involve partial credit scoring, which allows responses to be assigned credit according to differing degrees of "correctness" of responses. For each such item, a detailed coding guide that allows for full credit, partial credit or no credit is provided to persons trained in the coding of student responses across the range of participating countries to ensure coding of responses is done in a consistent and reliable way.

Reporting proficiency in mathematics

The outcomes of the PISA mathematics survey are reported in a number of ways. Estimates of overall mathematical proficiency are obtained for sampled students in each participating country, and a number of proficiency levels are defined. Descriptions of the degree of mathematical literacy typical of students in each level are also developed. In addition, aspects of overall mathematical proficiency are identified that will be of policy relevance to participating countries, separate estimates are obtained for students in relation to those aspects, and proficiency descriptions are also developed for the different levels defined on those scales. Aspects of potential use for reporting purposes can be defined in a variety of ways. For PISA 2003, scales based on the four broad content categories were developed. In Figure 1.3, descriptions for the six proficiency levels reported for the overall PISA mathematics scale in 2003, 2006 and 2009 are presented. These form the basis for the PISA 2012 mathematics scale.

■ Figure 1.3 ■

Proficiency scale descriptions for mathematics (2003-2009)

Level	
6	At Level 6 students can conceptualise, generalise and utilise information based on their investigations and modelling of complex problem situations. They can link different information sources and representations and flexibly translate among them. Students at this level are capable of advanced mathematical thinking and reasoning. These students can apply their insight and understandings along with a mastery of symbolic and formal mathematical operations and relationships to develop new approaches and strategies for attacking novel situations. Students at this level can formulate and precisely communicate their actions and reflections regarding their findings, interpretations, arguments and the appropriateness of these to the original situations.
5	At Level 5 students can develop and work with models for complex situations, identifying constraints and specifying assumptions. They can select, compare and evaluate appropriate problem-solving strategies for dealing with complex problems related to these models. Students at this level can work strategically using broad, well-developed thinking and reasoning skills, appropriate linked representations, symbolic and formal characterisations and insight pertaining to these situations. They can reflect on their actions and formulate and communicate their interpretations and reasoning.
4	At Level 4 students can work effectively with explicit models for complex concrete situations that may involve constraints or call for making assumptions. They can select and integrate different representations, including symbolic, linking them directly to aspects of real-world situations. Students at this level can utilise well-developed skills and reason flexibly, with some insight, in these contexts. They can construct and communicate explanations and arguments based on their interpretations, arguments and actions.
3	At Level 3 students can execute clearly described procedures, including those that require sequential decisions. They can select and apply simple problem-solving strategies. Students at this level can interpret and use representations based on different information sources and reason directly from them. They can develop short communications when reporting their interpretations, results and reasoning.
2	At Level 2 students can interpret and recognise situations in contexts that require no more than direct inference. They can extract relevant information from a single source and make use of a single representational mode. Students at this level can employ basic algorithms, formulae, procedures, or conventions. They are capable of direct reasoning and making literal interpretations of the results.
1	At Level 1 students can answer questions involving familiar contexts where all relevant information is present and the questions are clearly defined. They are able to identify information and to carry out routine procedures according to direct instructions in explicit situations. They can perform actions that are obvious and follow immediately from the given stimuli.

As well as the overall mathematics scale, three additional described proficiency scales are developed after the field trial and are then reported, based on the three mathematical processes described earlier – *formulating situations*

mathematically; employing mathematical concepts, facts, procedures, and reasoning; and *interpreting, applying and evaluating mathematical outcomes*.

Fundamental mathematical capabilities play a central role in defining what it means to be at different levels of the scales for mathematical literacy overall and for each of the reported processes – they define growing proficiency for all these aspects of mathematical literacy. For example, in the proficiency scale description for Level 4 (see Figure 1.3), the second sentence highlights aspects of mathematising and representation that are evident at this level. The final sentence highlights the characteristic communication, reasoning and argument of Level 4, providing a contrast with the short communications and lack of argument of Level 3 and the additional reflection of Level 5. The Box 1.1 "Fundamental mathematical capabilities and their relationship to item difficulty" at the end of this chapter describes the fundamental mathematical capabilities and the relationship each one has to development across levels of mathematical proficiency. In an earlier section of this framework and in Figure 1.2, each of the mathematical processes was described in terms of the fundamental mathematical capabilities that individuals might activate when engaging in that process.

For continuity with the reporting of outcomes of the 2003 survey when mathematics was last the major PISA survey domain and because of its usefulness for providing information for policy decisions, scales will also be reported based on the four content categories: *quantity, space and shape, change and relationships,* and *uncertainty and data*. These scales will continue to be interesting for countries since they can show profiles in aspects of mathematical proficiency resulting from specific curricular emphases.

Attitudes towards mathematics

Individuals' attitudes, beliefs and emotions play a significant role in their interest and response to mathematics in general, and their employment of mathematics in their individual lives. Students who feel more confident with mathematics, for example, are more likely than others to use mathematics in the various contexts that they encounter. Students who have positive emotions towards mathematics are in a position to learn mathematics better than students who feel anxiety towards that subject. Therefore, one goal of mathematics education is for students to develop attitudes, beliefs and emotions that make them more likely to successfully use the mathematics they know, and to learn more mathematics, for personal and social benefit.

The attention the PISA 2012 assessment of mathematics gives to these variables is based on claims that the development of positive attitudes, emotions and beliefs towards mathematics is in itself a valuable outcome of schooling and predisposes students to use mathematics in their lives; and also that such variables may contribute to explaining differences in the achievement of mathematical literacy. The PISA survey therefore includes items related to these variables. In addition, the PISA survey measures a range of background variables that enable the reporting and analysis of mathematical literacy for important subgroups of students (e.g. by gender, language or migration status).

To gather background information, students and the principals of their schools are asked to respond to background questionnaires of around 20 to 30 minutes in length. These questionnaires are central to the analysis and reporting of results in terms of a range of student and school characteristics.

Two broad areas of students' attitudes towards mathematics that dispose them to productive engagement in mathematics are identified as being of potential interest as an adjunct to the PISA 2012 mathematics assessment. These are students' interest in mathematics and their willingness to engage in it.

Interest in mathematics has components related to present and future activity. Relevant questions focus on students' interest in mathematics at school, whether they see it as useful in real life as well as their intentions to undertake further study in mathematics and to participate in mathematics-oriented careers. There is international concern about this area, because in many participating countries there is a decline in the percentage of students who are choosing mathematics-related future studies, whereas at the same time there is a growing need for graduates from these areas.

Students' willingness to do mathematics is concerned with the attitudes, emotions and self-related beliefs that dispose students to benefit, or prevent them from benefitting, from the mathematical literacy that they have achieved. Students who enjoy mathematical activity and feel confident to undertake it are more likely to use mathematics to think about the situations that they encounter in the various facets of their lives, inside and outside school. The constructs from the PISA survey that are relevant to this area include the emotions of enjoyment, confidence and (lack of) mathematics anxiety, and the self-related beliefs of self-concept and self-efficacy. A recent analysis of the subsequent progress of young Australians who scored poorly on PISA at age 15 found that those who "recognise the value of mathematics for their

future success are more likely to achieve this success, and that includes being happy with many aspects of their personal lives as well as their futures and careers" (Thomson and Hillman, 2010, p. 31). The study recommends that a focus on the practical applications of mathematics in everyday life may help improve the outlook for these low-achieving students.

The student questionnaire also includes sets of items related to *opportunity to learn*. There are items concerning student experience with applied mathematics problems of various types, student familiarity with mathematical concepts by name (including measures to guard against overclaiming) and student experience in class or tests with PISA style items. These measures will allow deeper analysis of the PISA results.

The results of the PISA 2012 survey will provide important information for educational policy makers in the participating countries about both the achievement-related and attitude-related outcomes of schooling. By combining information from the PISA assessment of mathematical literacy and the survey of attitudes, emotions and beliefs that predispose students to use their mathematical literacy, a more complete picture emerges.

Optional computer-based assessment of mathematics

PISA 2012 includes a computer-based assessment of mathematics.[4] While the computer-based assessment is optional for participating countries (given countries' varied technological capacities), there are two aspects to the rationale for including a computer-based mathematics assessment in PISA 2012. First, computers are now so commonly used in the workplace and in everyday life that a level of competency in mathematical literacy in the 21st century includes usage of computers (Hoyles et al., 2002). Computers now touch the lives of individuals around the world as they engage in their personal, societal, occupational and scientific endeavours. They offer tools for – among other things – computation, representation, visualisation, modification, exploration and experimentation on, of and with a large variety of mathematical objects, phenomena and processes. The definition of mathematical literacy for PISA 2012 recognises the important role of computer-based tools by noting that mathematically literate individuals are expected to use these in their endeavours to describe, explain, and predict phenomena. In this definition, the word "tool" refers to calculators and computers, as well as to other physical objects such as rulers and protractors used for measuring and construction. A second consideration is that the computer provides a range of opportunities for designers to write test items that are more interactive, authentic and engaging (Stacey and Wiliam, forthcoming). These opportunities include the ability to design new item formats (e.g. drag-and-drop), to present students with real-world data (such as a large, sortable dataset), or to use colour and graphics to make the assessment more engaging.

In response to these phenomena, an optional computer-based assessment of mathematics is a major area for innovation in the PISA 2012 assessment. Specially designed PISA units are presented on a computer, and students respond on the computer. They are also able to use pencil and paper to assist their thinking processes. Future PISA cycles may feature more sophisticated computer-based items, as developers and item writers become more fully immersed in computer-based assessment. Indeed, PISA 2012 represents only a starting point for the possibilities of the computer-based assessment of mathematics.

Making use of enhancements offered by computer technology results in assessment items that are more engaging to students, more colourful, and easier to understand. For example, students may be presented with a moving stimulus, representations of three-dimensional objects that can be rotated, or more flexible access to relevant information. New item formats, such as those calling for students to 'drag and drop' information or use 'hot spots' on an image, are designed to engage students, permit a wider range of response types and give a more rounded picture of mathematical literacy.

Investigations show that the mathematical demands of work increasingly occur in the presence of electronic technology so that mathematical literacy and computer use are melded together (Hoyles et al., 2002). For employees at all levels of the workplace, there is now an interdependency between mathematical literacy and the use of computer technology, and the computer-based component of the PISA survey provides opportunities to explore this relationship. A key challenge is to distinguish the mathematical demands of a PISA computer-based item from demands unrelated to mathematical proficiency, such as the information and communications technology (ICT) demands of the item, and new presentation formats. On the optional computer-based PISA 2012 survey it is important that the focus is on ensuring that the demand associated with the use of a tool in a particular item is significantly lower than the demand associated with the mathematics. Research has been conducted on the impact a computer-based testing environment has on students' performance (Bennett, 2003; Bennett et al., 2008; Mason et al., 2001; Richardson et al., 2002; Sandene et al., 2008), and the PISA 2012 survey provides an opportunity to further this knowledge, particularly to inform development of future computer-based tests for 2015 and beyond. By design, not all computer-based items will use new item formats, which might be helpful in monitoring the (positive or negative) impact that new item formats have on performance.

In order to establish control over the range of computer-based features of the test, for each item three aspects are described:

The mathematical competencies being tested: These comprise aspects of mathematical literacy applicable in any environment, not just computer environments, and are being tested in every computer-based assessment item.

Competencies that cover aspects of mathematics and ICT: These require knowledge of doing mathematics with the assistance of a computer or handheld device. These are being tested in some – but not all – computer-based assessment items. The computer-based test may include assessments of the following competencies:

- making a chart from data, including from a table of values (e.g. pie chart, bar chart, line graph) using simple 'wizards';
- producing graphs of functions and using the graphs to answer questions about the functions;
- sorting information and planning efficient sorting strategies;
- using hand-held or on-screen calculators;
- using virtual instruments such as an on-screen ruler or protractor; and
- transforming images using a dialog box or mouse to rotate, reflect or translate the image.

ICT skills: Just as pencil and paper assessments rely on a set of fundamental skills for working with printed materials, computer-based assessments rely on a set of fundamental skills for using computers. These include knowledge of basic hardware (e.g. keyboard and mouse) and basic conventions (e.g. arrows to move forward and specific buttons to press to execute commands). The intention is to keep such skills to a minimal core level in every computer-based assessment item.

SUMMARY

The aim of PISA with regard to mathematical literacy is to develop indicators that show how effectively countries are preparing students to use mathematics in every aspect of their personal, civic and professional lives, as part of their constructive, engaged and reflective citizenship. To achieve this, PISA has developed a definition of mathematical literacy and an assessment framework that reflects the important components of this definition. The mathematics assessment items developed and selected for inclusion in PISA 2012, based on this definition and framework, are intended to reflect a balance of relevant mathematical processes, mathematical content and contexts. These items are intended to determine how students can use what they have learnt. They call for students to use the content they know by engaging in processes and applying the capabilities they possess to solve problems that arise out of real-world experiences. The assessment provides problems in a variety of item formats with varying degrees of built-in guidance and structure, but the emphasis is on authentic problems where students must do the thinking themselves.

Box 1.1 **Fundamental mathematical capabilities and their relationship to item difficulty**

A good guide to the empirical difficulty of items can be obtained by considering which aspects of the fundamental mathematical capabilities are required for planning and executing a solution (Turner, 2012; Turner and Adams, 2012; Turner et al., forthcoming). The easiest items will require the activation of few capabilities and in a relatively straightforward way. The hardest items require complex activation of several capabilities. Predicting difficulty requires consideration of both the number of capabilities and the complexity of activation required. The sections below describe characteristics which make the activation of a single capability more or less complex (see also Turner, 2012).

Communication: Various factors determine the level and extent of the communication demand of a task, and the capability of an individual to meet these demands indicates the extent to which they possess the communication capability. For the receptive aspects of communication, these factors include the length and complexity of the text or other object to be read and interpreted, the familiarity of the ideas or information referred to in the text or object, the extent to which the information required needs to be disentangled from other information, the ordering of information and whether this matches the ordering of the thought processes required to interpret and use the information, and the extent to which there are different elements (such as text, graphic elements, graphs, tables, charts) that need to be interpreted in relation to each other. For the expressive aspects of communication, the lowest level of complexity is observed in tasks that demand simply provision of a numeric answer. As the requirement for a more extensive expression of a solution is added, for example when a verbal or written explanation or justification of the result is required, the communication demand increases.

Mathematising: In some tasks, mathematisation is not required – either the problem is already in a sufficiently mathematical form, or the relationship between the model and the situation it represents is not needed to solve the

...

problem. The demand for mathematisation arises in its least complex form when the problem solver needs to interpret and infer directly from a given model, or to translate directly from a situation into mathematics (e.g. to structure and conceptualise the situation in a relevant way, to identify and select relevant variables, collect relevant measurements, and/or make diagrams). The mathematisation demand increases with additional requirements to modify or use a given model to capture changed conditions or interpret inferred relationships; to choose a familiar model within limited and clearly articulated constraints; or to create a model where the required variables, relationships and constraints are explicit and clear. At an even higher level, the mathematisation demand is associated with the need to create or interpret a model in a situation where many assumptions, variables, relationships and constraints are to be identified or defined, and to check that the model satisfies the requirements of the task; or, to evaluate or compare models.

Representation: This mathematical capability is called on at the lowest level with the need to directly handle a given familiar representation, for example going directly from text to numbers, or reading a value directly from a graph or table. More cognitively demanding representation tasks call for the selection and interpretation of one standard or familiar representation in relation to a situation, and at a higher level of demand still when they require translating between or using two or more different representations together in relation to a situation, including modifying a representation; or when the demand is to devise a straightforward representation of a situation. Higher level cognitive demand is marked by the need to understand and use a non-standard representation that requires substantial decoding and interpretation; to devise a representation that captures the key aspects of a complex situation; or to compare or evaluate different representations.

Reasoning and argument: In tasks of very low demand for activation of this capability, the reasoning required may involve simply following the instructions given. At a slightly higher level of demand, items require some reflection to connect different pieces of information in order to make inferences (e.g. to link separate components present in the problem, or to use direct reasoning within one aspect of the problem). At a higher level, tasks call for the analysis of information in order to follow or create a multi-step argument or to connect several variables; or to reason from linked information sources. At an even higher level of demand, there is a need to synthesise and evaluate information, to use or create chains of reasoning to justify inferences, or to make generalisations drawing on and combining multiple elements of information in a sustained and directed way.

Devising strategies: In tasks with a relatively low demand for this capability, it is often sufficient to take direct actions, where the strategy needed is stated or obvious. At a slightly higher level of demand, there may be a need to decide on a suitable strategy that uses the relevant given information to reach a conclusion. Cognitive demand is further heightened with the need to devise and construct a strategy to transform given information to reach a conclusion. Even more demanding tasks call for the construction of an elaborated strategy to find an exhaustive solution or a generalised conclusion; or to evaluate or compare different possible strategies.

Using symbolic, formal and technical language and operations: The demand for activation of this capability varies enormously across tasks. In the simplest tasks, no mathematical rules or symbolic expressions need to be activated beyond fundamental arithmetic calculations, operating with small or easily tractable numbers. Work with more demanding tasks may involve sequential arithmetic calculations or direct use of a simple functional relationship, either implicit or explicit (e.g. familiar linear relationships); use of formal mathematical symbols (e.g. by direct substitution or sustained arithmetic calculations involving fractions and decimals); or an activation and direct use of a formal mathematical definition, convention or symbolic concept. Further increased cognitive demand is characterised by the need for explicit use and manipulation of symbols (e.g. by algebraically rearranging a formula), or by activation and use of mathematical rules, definitions, conventions, procedures or formulas using a combination of multiple relationships or symbolic concepts. A yet higher level of demand is characterised by the need for a multi-step application of formal mathematical procedures, working flexibly with functional or involved algebraic relationships, or using both mathematical technique and knowledge to produce results.

Using mathematical tools: Tasks and activities involving a relatively low level of demand for this capability may require direct use of familiar tools, such as a measuring instrument, in situations where use of those tools is well-practised. Higher levels of demand arise when using the tool involves a sequence of processes, or linking different information using the tool, and when familiarity of the tools themselves is lower or when the situation in which the application of the tool is required is less familiar. Further increased demand is seen when the tool is to be used to process and relate multiple data elements, when the application of a tool is needed in a situation quite different from familiar applications, when the tool itself is complex with multiple affordances, and when there is a need for reflection to understand and evaluate the merits and limitations of the tool.

ILLUSTRATIVE PISA MATHEMATICS ITEMS

The following released PISA items are intended to illustrate relevant aspects and nuances of the PISA 2012 framework. The seven items were selected to represent a spread across item type, process, content and context, as well as to describe the activation of the fundamental mathematical capabilities, but they are not intended to represent the full range of any particular aspect.

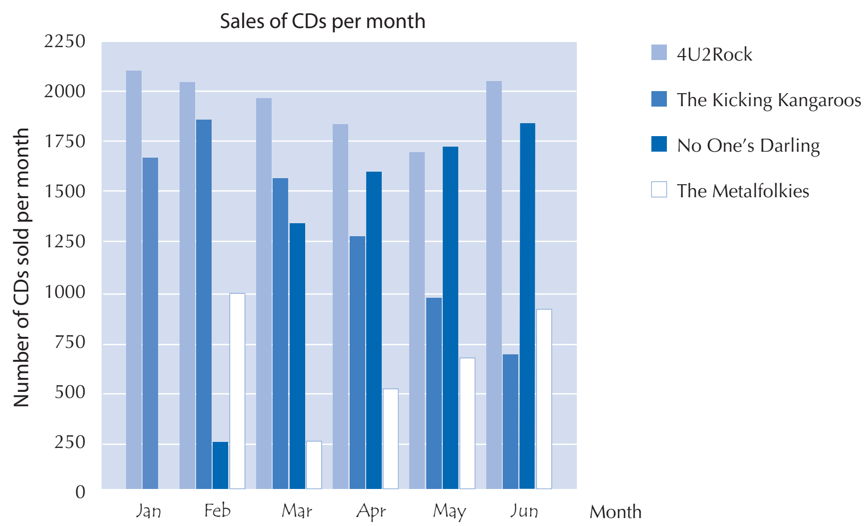
CHARTS

The first illustrative unit is titled *CHARTS*. It comprises stimulus information in the form of text and a bar graph that represents music CD sales for four bands over a period of six months, and three simple multiple choice items (Figure 1.4).

■ Figure 1.4 ■

Items for the unit *CHARTS*

In January, the new CDs of the bands *4U2Rock* and *The Kicking Kangaroos* were released. In February, the CDs of the bands *No One's Darling* and *The Metalfolkies* followed. The following graph shows the sales of the bands' CDs from January to June.



QUESTION 1

How many CDs did the band *The Metalfolkies* sell in April?

A. 250

B. 500

C. 1 000

D. 1 270

QUESTION 2

In which month did the band *No One's Darling* sell more CDs than the band *The Kicking Kangaroos* for the first time?

A. No month

B. March

C. April

D. May

QUESTION 3

The manager of The *Kicking Kangaroos* is worried because the number of their CDs that sold decreased from February to June.

What is the estimate of their sales volume for July if the same negative trend continues?

A. 70 CDs

B. 370 CDs

C. 670 CDs

D. 1 340 CDs

In preparing national versions, PISA countries were expected to replace the band names with fictitious names suitable for their local context.

CHARTS was used in the PISA 2012 main survey. The three items of *CHARTS* each lie in the *uncertainty and data* content category, since they ask students to read, interpret and use data presented in a mathematical graphical form. They each lie in the *societal* context category, since the data relate to public information about music sales, the kind that might be found in a newspaper, music magazine or on line. The first two questions are examples of the *interpreting, applying and evaluating mathematical outcomes* process category, since these questions involve interpreting the mathematical information presented in the chart in relation to context features represented; while the third question fits the *employing mathematical facts, concepts, procedures and reasoning* category because its focus is on applying procedural knowledge to manipulate the mathematical representation in order to make a further inference. The three questions were among the easiest questions used in the PISA 2012 main survey.

Question 1, shown in Figure 1.4, calls for a straight-forward reading of data from the graph to answer a question about the context. Students needed to orient themselves to the information presented, identify which data series represents sales for the specified band, which bar represents the specified month within that series, and read the value 500 CDs directly from the vertical axis. The text is simple and clear, creating a very low *communication* demand. The *strategy* required is straightforward: simply to find the specified information in the graph. The *mathematising* demand is to make an inference about the sales situation directly from the graphical model. The *representation* capability is called on at a low level, involving reading a value directly from the graph. The graph format would be familiar to most 15-year-olds, and with effort required only to read the labels to identify what is represented. One axis of the graph is a category axis (months) and the height of the relevant bar is labelled (500) so no understanding of scale is required. The *technical* knowledge required is minimal beyond familiarity with the graph form; and only a direct inference is required, hence very low level demand for *reasoning and argument*. This was an extremely easy item, with some 87% of students identifying the correct response, B.

Question 2 is only slightly more difficult, with about 78% correctly identifying response C. To answer this question, students must observe the relationship between two data series displayed in the bar chart, taking notice of how that relationship changes over the time period shown, in order to recognise that the condition specified in the question was first met in April.

The *communication* demand is similar to that for Question 1. The *strategy* needed is slightly more involved, since multiple elements of the two data series need to be drawn together. The *mathematisation* required again involves making an inference about the sales situation fairly directly from the graph. The *representation* demand is slightly raised from the requirement to read a single data point in Question 1, involving the linking of two data series and the time variable. The demand for *using symbolic, formal and technical language and operations* remains low as only a qualitative comparison is required; and the *reasoning and argument* demand is slightly elevated since a small sequence of reasoning steps is required.

Question 3 is somewhat different from the first two, in that the main focus is on understanding a mathematical relationship depicted in the graph, and extrapolating that relationship to predict the next monthly value. The link to the context is still there, but the main demand is to work with the mathematical information shown. One way to do this would be to read the monthly data values for the series in question, estimate a reasonable average value by which each monthly value is reducing, and apply that same reduction to the data value given for the final month shown. The *communication* demand remains low. The main challenge is to avoid the distraction of the data series of other bands. However, the only common wrong answer was perhaps due to an error in understanding the phrase "the same negative trend". Overall, 15% of students answered C, estimating the sales for July to be equal to the sales for June. They may have chosen the constant value because it maintained the same bad June sales figures into July. The *strategy* needed is clearly more involved than in the first two questions and its implementation requires some monitoring. There are decisions to make, such as whether to use all five February to June data points for this band, or to use the average change from February to June, and whether to calculate exactly, to draw or visualise a trend line or to work with broad estimates noting that each month the sales drop by just over one vertical scale division. The *mathematisation* demand involves a small manipulation of the given model in relation to the context; some calculation is required (repeated subtraction of multi-digit numbers, scale reading between labelled points) that would add to the demand for *using symbolic, formal and technical language and operations*. The *representation* demand involves inferring a trend relationship depicted in the graph; and a small sequence of *reasoning* steps is required to solve the problem. Nevertheless, this item is also relatively easy, with some 76% of students selecting the correct response B in the PISA 2012 main survey administration.

■ Figure 1.5 ■
Items for the unit *CLIMBING MOUNT FUJI*

CLIMBING MOUNT FUJI

Mount Fuji is a famous dormant volcano in Japan.

QUESTION 1

Mount Fuji is only open to the public for climbing from 1 July to 27 August each year. About 200 000 people climb Mount Fuji during this time.

On average, about how many people climb Mount Fuji each day?

A. 340

B. 710

C. 3 400

D. 7 100

E. 7 400

QUESTION 2

The Gotemba walking trail up Mount Fuji is about 9 kilometres (km) long.

Walkers need to return from the 18 km walk by 8 pm.

Toshi estimates that he can walk up the mountain at 1.5 kilometres per hour on average, and down at twice that speed. These speeds take into account meal breaks and rest times.

Using Toshi's estimated speeds, what is the latest time he can begin his walk so that he can return by 8 pm?

....

QUESTION 3

Toshi wore a pedometer to count his steps on his walk along the Gotemba trail.

His pedometer showed that he walked 22 500 steps on the way up.

Estimate Toshi's average step length for his walk up the 9 km Gotemba trail. Give your answer in centimetres (cm).

Answer .... cm

CLIMBING MOUNT FUJI

A second illustrative unit is titled *CLIMBING MOUNT FUJI*, shown in Figure 1.5. The first question is a simple multiple choice item and the second and third questions are constructed response items requiring numerical answers. The third item has partial credit available. This is used for a small proportion of PISA items where qualitatively different kinds of response can be given, and where markedly different abilities can be associated with different kinds of responses.

CLIMBING MOUNT FUJI was used in the PISA 2012 main survey, and then released into the public domain. Questions 1 and 3 lie in the *quantity* content category, since they ask students to calculate with dates and measurements and make conversions. Question 2 has speed as its central concept and is therefore in the *change and relationships* content category.

They each lie in the *societal* context category, since the data relate to information about public access to Mount Fuji and its trails. The first two questions are examples of the *formulating situations mathematically* process category, since the main demand of these questions involves creating a mathematical model that can answer the posed questions.

Question 3 is placed in the *employing mathematical facts, concepts, procedures and reasoning* category because the main demand here is to calculate an average, taking care to convert units appropriately, hence working essentially within the mathematical details of the problem rather than connecting those details with the contextual elements. The three questions were of varying difficulty in the PISA 2012 main survey. Question 1 was of medium difficulty, and Questions 2 and 3 were both very difficult.

Question 1 calls for calculation of the average number of people per day. The text is simple and clear, creating a low *communication* demand. The *strategy* required is of moderate demand, because it involves finding the number of days from the dates provided and using this to find the average. This multiple step solution requires some monitoring, which is also part of the *devising strategies* demand. The *mathematising* demand is very low, because the mathematical quantities required are directly given in the question (number of people per day). Demand for the *representation* capability is similarly low – only numerical information and text are involved. The *technical* knowledge required includes knowing how to find an average, being able to calculate number of days from dates, being able to perform the division (using calculator or not, depending on country assessment policy), and rounding the result appropriately. There is low level demand for *reasoning and argument*. This was an item of medium difficulty, with some 46% of students in the PISA 2012 main survey administration identifying the correct response, C. The two most popular wrong choices were E (which is obtained by using 27 days instead of 31+27 days) with 19% of responses; and A (a place value error) with 12% of responses.

Question 2 is considerably more difficult, with about 12% correct in the PISA 2012 main survey. One factor in this difficulty is that it is a *constructed response* item, rather than *selected response*, so students are given no guidance regarding possible answers, but there are many other factors. About 61% of responses to the PISA 2012 survey administration of this question were wrong answers, not missing.

The *communication* demand is low and, in its receptive aspects, similar to that for Question 1. The constructive communication only requires a numerical response. The *strategy* needed is much more involved, since a plan with three main parts needs to be assembled. Times up and down the mountain need to be calculated from the average speeds, and then the starting time needs to be calculated from the finishing time and the time the walk takes. The *mathematisation* required is moderately high, involving aspects such as understanding how meal times are already included and even that the trail will first be up and then separately down. The *representation* demand is minimal, with only the interpretation of text required. The demand for *using symbolic, formal and technical language and operations* is moderately high: all the calculations are relatively simple (although division by the decimal 1.5 km per hour may be challenging) but it requires sustained accuracy, and the formula for time from speed and distance is required either implicitly or explicitly. The *reasoning and argument* demand is also moderately high.

Question 3 is also quite difficult. The main focus is to calculate average step length from distance and number of steps, with unit conversions required. For this item, 11% of responses in the PISA 2012 survey administration gained full credit for the correct response 40 cm, and a further 4% gained partial credit for responses such as 0.4 (the answer left in metres) or 4 000 where an incorrect conversion factor from metres to centimetres has probably been used. For the PISA 2012 main survey administration, 62% of responses were incorrect answers, not missing. The *communication* demand remains low as with the earlier questions, since the text is fairly clear and easy to interpret and the requirement for a single number as a sufficient answer. The *strategy* needed for Question 3 is similar to that for Question 1 – both require finding an average. Although both use similar models to find ‘averages’, the *reasoning and argument* needed for Question 3 is more involved than for Question 1. In Question 1, the quantity required is “people per day” where the number of people is given and the number of days is readily calculated. Question 3 requires “step length” to be calculated from a total distance and a total number of steps. More reasoning is required to link these quantities in Question 3 (for example linking the given distance with the length). The *mathematisation* demand is similarly higher in Question 3, understanding how the real world quantity of step length relates to the overall measures. An appreciation of the real world context, including that step length is likely to be around 50 cm (rather than 500 cm or 0.5 cm), is also useful for monitoring the reasonableness of the answer. The demand for *using symbolic, formal and technical language and operations* is moderately high, because of the division of a small number (9 km) by a large number (22 500 steps) and the need for using known conversion factors. The *representation* demand is again low, since only text is involved.

PIZZAS

The open constructed-response item *PIZZAS* shown in Figure 1.6 is simple in form, yet rich in content, and illustrates various elements of the mathematics framework. It was initially used in the first PISA field trial in 1999, then was released for illustrative purposes and has appeared as a sample item in each version of the PISA mathematics framework published since 2003. This was one of the most difficult items used in the 1999 field trial item pool, with only 11% correct.

PIZZAS is set in a *personal* context with which many 15-year-olds would be familiar. The context category is personal since the question posed is which pizza provides the purchaser with the better value for the money. It presents a relatively low reading demand, thereby ensuring the efforts of the reader can be directed almost entirely to the underlying mathematical intentions of the task.

■ Figure 1.6 ■
Item for the unit *PIZZAS*

PIZZAS

A pizzeria serves two round pizzas of the same thickness in different sizes. The smaller one has a diameter of 30 cm and costs 30 zeds. The larger one has a diameter of 40 cm and costs 40 zeds.

Which pizza is better value for money? Show your reasoning.

Every day terms from the real world must be interpreted mathematically (*round, same thickness, different sizes*). The size variable is given mathematical definition in the diameters provided for the two pizzas. The costs are provided in the neutral currency *zeds*. Size and cost are linked through the concept of *value for money*.

The item draws on several areas of mathematics. It has geometrical elements that would normally be classified as part of the *space and shape* content category. The pizzas can be modelled as thin circular cylinders, so the area of a circle is needed. The question also involves the *quantity* content category with the implicit need to compare the quantity of pizza to amount of money. However, the key to this problem lies in the conceptualisation of the relationships among properties of the pizzas, and how the relevant properties change from the smaller pizza to the larger one. Because those aspects are at the heart of the problem, this item is categorised as belonging to the *change and relationships* content category.

The item belongs to the *formulating* process category. A key step to solving this problem, indeed the major cognitive demand, is to formulate a mathematical model that encapsulates the concept of *value for money*. The problem solver must recognise that because pizzas ideally have uniform thickness and the thicknesses are the same, the focus of analysis can be on the area of the circular surface of the pizza instead of volume or mass. The relationship between amount of pizza and amount of money is then captured in the concept of *value for money* modelled as 'cost per unit of area'. Variations such as area per unit cost are also possible. Within the mathematical world, *value for money* can then be calculated directly and compared for the two circles, and is a smaller quantity for the larger circle. The real world interpretation is that the larger pizza represents better value for money.

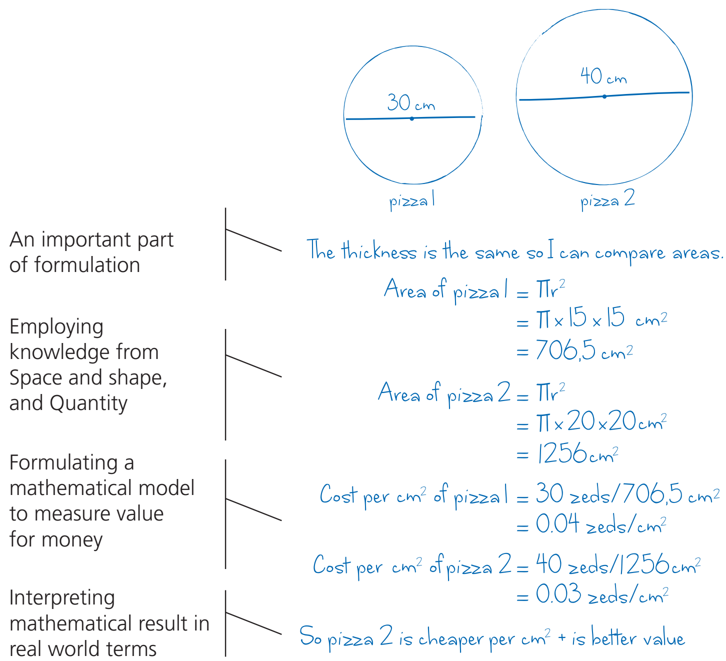
An alternative form of reasoning, which reveals even more clearly the item's classification in *change and relationships*, would be to say (explicitly or implicitly) that the area of a circle increases in proportion to the square of the diameter, so has increased in the ratio of $(4/3)^2$, while the cost has only increased in the proportion of (4/3). Since $(4/3)^2$ is greater than (4/3), the larger pizza is better value.

While the primary demand and the key to solving this problem comes from formulating, placing this item in the *formulating situations mathematically* process category, aspects of the other two mathematical process are also apparent in this item. The mathematical model, once formulated, must then be employed effectively, with the application of appropriate reasoning along with the use of appropriate mathematical knowledge and area and rate calculations. The result must then be interpreted properly in relation to the original question.

The solution process for *PIZZAS* demands the activation of the fundamental mathematical capabilities to varying degrees. *Communication* comes in to play at a relatively low level in reading and interpreting the rather straight-forward text of the problem, and is called on at a higher level with the need to present and explain the solution. The need to *mathematise* the situation is a key demand of the problem, specifically the need to formulate a model that captures *value for money*. The problem solver must devise a *representation* of relevant aspects of the problem, including the symbolic representation of the formula for calculating area, and the expression of rates that represent *value for money*, in order to develop a solution. The *reasoning* demands (for example, to decide that the thickness can be ignored, and justifying the approach taken and the results obtained) are significant, and the need for *devising strategies* to control the calculation and modelling processes required is also a notable demand for this problem. *Using symbolic, formal and technical language and operations* comes into play with the conceptual, factual and procedural knowledge required to process the circle geometry, and the calculations of the rates. *Using mathematical tools* is evident at a relatively low level if students use a calculator efficiently.

In Figure 1.7, a sample student response to the *PIZZAS* item is presented, to further illustrate the framework constructs. A response like this would be awarded full credit.

■ Figure 1.7 ■
Sample response to *PIZZAS*



LITTER

The item *LITTER* shown in Figure 1.8 is also presented to illustrate aspects of the mathematics framework. This constructed-response item was used in the PISA 2003 main survey and then released into the public domain. The average percent correct for this item in OECD countries was slightly over 51%, placing it near to the middle of the item pool in difficulty.

■ Figure 1.8 ■
Item for the unit *LITTER*

For a homework assignment on the environment, students collected information on the decomposition time of several types of litter that people throw away:

Type of litter	Decomposition time
Banana peel	1-3 years
Orange peel	1-3 years
Cardboard boxes	0.5 years
Chewing gum	20-25 years
Newspapers	A few days
Polystyrene cups	Over 100 years

A student thinks of displaying the results in a bar graph.

*Give **one** reason why a bar graph is unsuitable for displaying these data.*

This item is set in a *scientific* context, since it deals with data of a scientific nature (decomposition time). The mathematical content category is *uncertainty and data*, since it primarily relates to the interpretation and presentation of data, although *quantity* is involved in the implicit demand to appreciate the relative sizes of the time intervals involved. The mathematical process category is *interpreting, applying and evaluating mathematical outcomes* since the focus is on evaluating the effectiveness of the mathematical outcome (in this case an imagined or sketched bar graph) in portraying the data about the real world contextual elements. The item involves reasoning about the data presented, thinking

mathematically about the relationship between the data and their presentation, and evaluating the result. The problem solver must recognise that these data would be difficult to present well in a bar graph for one of two reasons: either because of the wide range of decomposition times for some categories of litter (this range cannot readily be displayed on a standard bar graph), or because of the extreme variation in the time variable across the litter types (so that on a time axis that allows for the longest period, the shortest periods would be invisible). Student responses such as those reproduced below have been awarded credit for this item.

RESPONSE 1

"Because it would be hard to do in a bar graph because there are 1-3, 1-3, 0.5, etc. so it would be hard to do it exactly."

RESPONSE 2

"Because there is a large difference from the highest sum to the lowest therefore it would be hard to be accurate with 100 years and a few days."

The solution process for *LITTER* demands the activation of the fundamental mathematical capabilities as follows. *Communication* comes in to play with the need to read the text and interpret the table, and is also called on at a higher level with the need to answer with brief written reasoning. The demand to *mathematise* the situation arises at a low level with the need to identify and extract key mathematical characteristics of a bar graph as each type of litter is considered. The problem solver must interpret a simple tabular *representation* of data, and must imagine a graphical representation, and linking these two representations is a key demand of the item. The *reasoning* demands of the problem are at a relatively low level, as is the need for *devising strategies*. *Using symbolic, formal and technical language and operations* comes into play with the procedural and factual knowledge required to imagine construction of bar graphs or to make a quick sketch, and particularly with the understanding of scale needed to imagine the vertical axis. *Using mathematical tools* is likely not needed.

ROCK CONCERT

A further illustrative item, *ROCK CONCERT*, is presented in Figure 1.9. This selected-response item (here simple multiple choice) was used in the field trial prior to the PISA 2003 survey, then was released into the public domain for illustrative purposes. About 28% of sampled students got this item correct (choice C), making it a moderately difficult item relative to the pool of items used in the field trial. *ROCK CONCERT* is set in a societal context, because the item is set at the level of the rock concert organisation, even though it draws on personal experience of being in crowds. It is classified within the *quantity* content category because of the numerical calculation required, though it also has some elements that relate to the *space and shape* category.

■ Figure 1.9 ■
Item for the unit *ROCK CONCERT*

For a rock concert, a rectangular field of size 100 m by 50 m was reserved for the audience. The concert was completely sold out and the field was full with all the fans standing.

Which one of the following is likely to be the best estimate of the total number of people attending the concert?

A. 2 000
B. 5 000
C. 20 000
D. 50 000
E. 100 000

This item calls on each of the three process categories but the primary demand comes from *formulating situations mathematically*, with the need to make sense of the contextual information provided (the field size and shape; the rock concert is full; fans are standing) and translate it into a useful mathematical form. There is also the need to identify information that is missing, but that could reasonably be estimated based on real-life knowledge and assumptions. Specifically there is a need to devise a model for the space required for an individual fan or a group of fans. Working within mathematics, the problem solver needs to *employ mathematical concepts, facts, procedures and reasoning* to link the area of the field and the area occupied by a fan to the number of fans, making the quantitative comparisons needed. And *interpreting, applying and evaluating mathematical outcomes* is required to check the reasonableness of the solution, or to evaluate the answer options against the mathematical results of calculations performed.

An alternative model is to imagine the fans standing uniformly in equal rows across the field and to estimate the number of fans by multiplying the estimated number of rows by the estimated number of fans in each row. Problem solvers with strong skills in formulating mathematical models may appreciate the effectiveness of this rows-and-columns model, despite the stark contrast between it and the behaviour of fans at a rock concert. The correct answer is insensitive to which of several reasonable models is being adopted by the student.

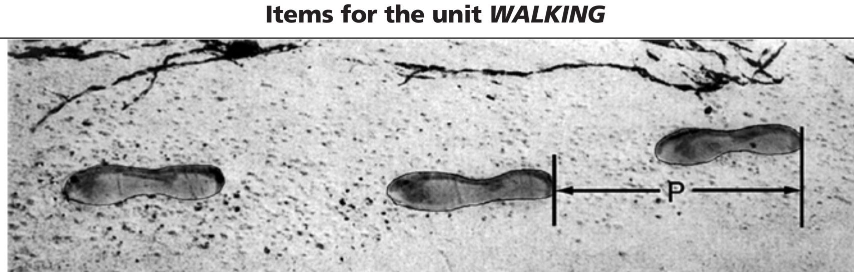
The fundamental mathematical capabilities come into play for this question in the following ways. *Communication* is called on at a relatively low level with the need to read and understand the text. The mathematical importance of words such as *rectangular and size*, the phrase the *field was full*, and the instruction to *estimate*, must all be interpreted and understood. Some real-world knowledge will help to do this. The task has a significant *mathematisation* demand, since solving the problem would require making certain assumptions about the space that a person might occupy while standing as well as requiring the creation of a basic model such as (number of fans) x (average space for a fan) = (area of field). To do this one must *represent* the situation mentally or diagrammatically, as part of formulating the model to link the space for a fan with the area of the field. *Devising a strategy* comes into the process of solving this problem at several stages, such as when deciding on how the problem should be approached, when imagining what kind of model could be useful to capture the space occupied by a fan at the concert, and when recognising the need for some checking and validation procedures. One solution strategy would involve postulating an area for each person, multiplying it by the number of people given in each of the options provided, and comparing the result to the conditions given in the question. Alternatively, the reverse could be done, starting with the area provided and working backwards using each of the response options to calculate the corresponding space per person, and deciding which one best fits the criteria established in the question. *Using symbolic, formal and technical language and operations* comes into play in implementing whatever strategy was adopted, by interpreting and using the dimensions provided and in carrying out the calculations required to relate the field area to the area for an individual. *Reasoning and argument* would come in to play with the need to think clearly about the relationship between the model devised, the resulting solution, and the real context, in order to validate the model used and to check that the correct answer is chosen. *Using mathematical tools* is unlikely to be needed.

WALKING

The PISA unit *WALKING*, presented in Figure 1.10, shows a somewhat counter-intuitive but well-established algebraic relationship between two variables, based on the observation of a large number of men walking at a natural pace, and asks students two questions that demand activation of algebraic knowledge and skills. For the second question, strategic thinking, reasoning and argument capabilities are also demanded at a level that challenges many 15-year-olds. These items were used in the PISA 2003 main survey, then released into the public domain and have subsequently been used as illustrative items in the PISA 2009 framework and in other publications. Both questions require students to work with the information given and to construct their response. Both items fit within the same framework categories: the *change and relationships* content category, since they relate to the relationships among variables, in this case expressed in algebraic form; the *personal* context category, since they focus on matters relating directly to the experience and perspective of the individual; and the *employing mathematical facts, concepts, procedures and reasoning* process category, since the problems have been expressed in terms that already have mathematical structure, and the work required is largely intra-mathematical manipulation of mathematical concepts and objects.

■ Figure 1.10 ■

Items for the unit *WALKING*



The picture shows the footprints of a man walking. The pacelength P is the distance between the rear of two consecutive footprints.

For men, the formula $\frac{n}{P} = 140$ gives an approximate relationship between n and P where:

n = number of steps per minute, and

P = pacelength in metres.

QUESTION 1

If the formula applies to Heiko's walking and Heiko takes 70 steps per minute, what is Heiko's pacelength? Show your work.

....

....

QUESTION 2

Bernard knows his pacelength is 0.80 metres. The formula applies to Bernard's walking.

Calculate Bernard's walking speed in metres per minute and in kilometres per hour. Show your working out.

....

....

Question 1 had an international percentage correct figure of 36% in the 2003 main survey, making it more difficult than about 70% of items in the 2003 pool. This is surprising, since mathematically all that is required is to substitute the value $n=70$ into the formula, and implement some reasonably straight-forward algebraic manipulation of the formula to find the value of P. This item illustrates the observation that has frequently been made about PISA survey items that when test questions are placed in some real world context, even when the mathematical components are presented clearly in the question, 15-year-old students often struggle to apply their mathematical knowledge and skills effectively.

The fundamental mathematical capabilities come into play for this question in the following ways. *Communication* is called on with the need to read and understand the stimulus, and later to articulate a solution and show the work involved. The task has no real *mathematisation* demand, since a mathematical model is provided in a form that would be familiar to many 15-year-old students. The *representation* demand is significant, given that the stimulus includes a graphic element, text and an algebraic expression that must be related to each other. *Devising a strategy* comes into the solution process at a very low level, since the strategy needed is very clearly expressed in the question. Minimal *reasoning and argument* is needed, again because the task is clearly stated and all required elements are obvious. *Using symbolic, formal and technical language and operations* comes into play in performing the substitution and manipulating the expression to make P the subject of the equation.

Question 2 is more difficult, with an international average percentage correct of 20%, meaning it was among the most difficult 10% of items used in the 2003 PISA survey. *Devising a strategy* for this question is complex because of the number of steps involved, and the resulting need to keep focused on the desired endpoint: P is known and so n can be found from the given equation; multiplying n by P gives the speed in number of metres travelled per minute; then proportional reasoning can be used to change the units of speed to kilometres per hour. Three levels of credit were available to accommodate solutions for which only partial progress towards a complete solution was achieved. The difference in the observed percentage correct for Question 2 compared to Question 1 can probably best be explained by describing the different activation of the fundamental mathematical capabilities that are required. The *communication* required for the two questions is comparable at the stage of reading and understanding the question, but in Question 2 the diagram has to be used to explicitly link one step and the given pacelength, a relationship not needed in Question 1, and the presentation of the solution demands higher level expressive communication skills for Question 2. The task has a new *mathematisation* demand, since solving the problem would require devising a proportional model for Bernard's walking speed in the units requested. Such a solution process requires activation of effective and sustained control mechanisms across a multi-step procedure, hence the *devising a strategy* capability is required at a much higher level than was the case for Question 1. The *representation* demands in the second question go beyond those needed for Question 1 with the need to work more actively with the given algebraic representation. Implementing the strategy devised and using the representations identified involves *using symbolic, formal and technical language and operations* that includes the algebraic manipulations, and the application of proportions and arithmetic calculations to perform the required conversions. *Reasoning and argument* comes in to play throughout with the sustained and connected thought processes required to proceed with the solution. *Using mathematical tools* is evident at a relatively low level if students use a calculator efficiently.

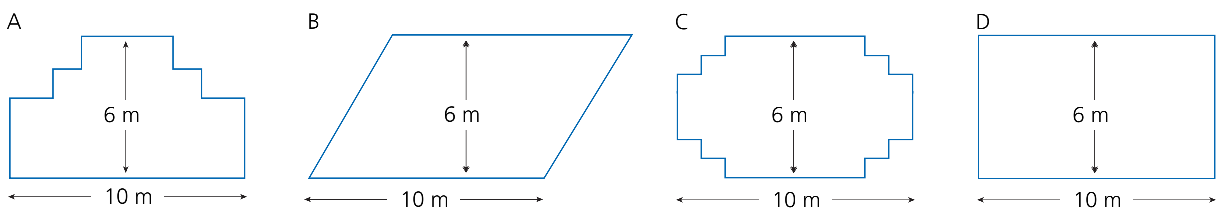
CARPENTER

The PISA item *CARPENTER* is presented in Figure 1.11. This item was used in both the PISA 2000 and 2003 surveys, and then released into the public domain. It illustrates a form of selected-response item known as the complex multiple-choice format, for which students must select one response from options attached to each of a number of statements or questions. In this case, students gained full credit by correctly identifying that all designs except Design B can be made with the specified amount of timber.

The item fits into the *space and shape* content category, since it deals with properties of shapes. It is associated with the *occupational* context category as it deals with a work task of a carpenter. The item is classified under the *employing mathematical concepts, facts, procedures and reasoning* process category, since most of the work involves applying procedural knowledge to well defined mathematical objects; although it also involves some degree of *interpreting, applying and evaluating mathematical outcomes* given the need to link the mathematical objects represented to the contextual element – the constraint imposed by the available timber.

■ Figure 1.11 ■
Item for the unit *CARPENTER*

A carpenter has 32 metres of timber and wants to make a border around a garden bed. He is considering the following designs for the garden bed.



Circle either "Yes" or "No" for each design to indicate whether the garden bed can be made with 32 metres of timber.

Garden bed design	Using this design, can the garden bed be made with 32 metres of timber?
Design A	Yes / No
Design B	Yes / No
Design C	Yes / No
Design D	Yes / No

This was one of the more difficult items in the PISA 2003 survey, with a correct response rate of a little less than 20%. It can be solved by the application of geometrical knowledge and reasoning. Enough information is given to enable direct calculation of the exact perimeter for Designs A, C and D, each of which is 32 metres. However, insufficient information is given for Design B; therefore a different approach is required. It can be reasoned that while the 'horizontal' components of the four shapes are equivalent, the oblique sides of Design B are longer than the sum of the 'vertical' components of each of the other shapes.

The *communication* capability is called on in reading and understanding the question, and to link the information provided in the text with the graphical *representation* of the four garden beds. The task has been presented in overtly mathematical form, hence no *mathematisation* is needed. Real world considerations, such as the lengths of the pieces of timber available and the geometry of the corners, do not come into the problems as posed here. The key capability demanded to solve the problem is the *reasoning and argument* needed to identify Design B which has too great a perimeter, and to appreciate that the lengths of the 'vertical' components of Design A are in themselves unknown, but that the total 'vertical' length is known (similarly with Design C with both vertical and horizontal lengths). *Devising a strategy* involves recognising that the perimeter information needed can be found in spite of the fact that some of the individual lengths are not known. *Using symbolic, formal and technical language and operations* is needed in the form of an understanding and manipulating of the perimeter of the shapes presented, including both the properties of the sides, and the addition of the side lengths. *Using mathematical* tools is likely not needed.

Notes

1. In some countries, "mathematical tools" can also refer to established mathematical procedures such as algorithms. For the purposes of the PISA framework, "mathematical tools" refers only to the physical and digital tools described in this section.

2. The standards for two sets of countries were analysed. The sets were nine OECD countries (Australia [New South Wales], [Flemish] Belgium, Canada [Alberta], Finland, Ireland, Japan, Korea, New Zealand and the United Kingdom), and six high-performing countries ([Flemish] Belgium, Canada [Alberta], Chinese Taipei, Finland, Korea, and Singapore). A constraint of the analysis was that standards had to be available in English.

3. Those familiar with earlier frameworks will note that the category *uncertainty* is now called *uncertainty and data*. This name change is intended only to describe the category more clearly, and is not a fundamental change to the category itself.

4. In 2006, PISA pilot tested a computer-based science assessment, and in 2009 included an optional digital reading assessment.

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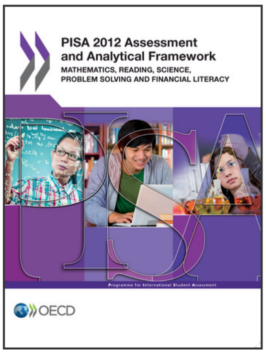
PISA 2012 Assessment
and Analytical Framework
MATHEMATICS, READING, SCIENCE,
PROBLEM SOLVING AND FINANCIAL LITERACY
OECD